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Confluence by decreasing diagrams

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Abstract

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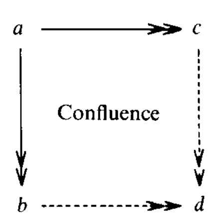
We present a confluence criterion, local decreasingness, for abstract reduction systems. This criterion is shown to be a considerable generalisation of several well-known confluence criteria.

1. Introduction

An abstract reduction system is a set of objects equipped with some binary "reduction" relations. As they have so little structure, abstract reduction systems can be viewed as abstractions of several kinds of rewriting such as string rewriting, term rewriting and graph rewriting. In the case of term rewriting, the objects model terms and reduction relations model (nondeterministic) computations.

A desirable property in computing is that results of computations are unique (if they exist). In the case that whenever we have two "diverging" computations starting from the same term, a common result can be reached by "converging" computations (the so-called confluence or Church–Rosser property), uniqueness is guaranteed.



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In this paper we present a confluence theorem that subsumes a number of classical confluence lemmata. A typical way to check confluence is to investigate how reduction steps interact. The idea is that this can be expressed abstractly by grading reduction steps with an ordered set of labels. Reduction sequences can then be graded with certain multisets of labels, ordered by the standard multiset exstension of the label order. A divergence $b \twoheadleftarrow a \twoheadrightarrow c$ is graded by the multiset union of the grades of the reduction sequences $a \twoheadrightarrow b$ and $a \twoheadrightarrow c$. A confluence diagram is graded by its divergence and is said to be decreasing if the measures of the convergent reductions $a \twoheadrightarrow b \twoheadrightarrow d$ and $a \twoheadrightarrow c \twoheadrightarrow d$ are both less than or equal to the measure of the diagram.

We define the measure of a reduction sequence to be the multiset of the lexicographically maximal step labels of the sequence (step labels not less than the label of an earlier step). Decreasing diagrams can then be pasted to yield decreasing diagrams. The main theorem states that if the label order is well-founded and every local confluence diagram is decreasing, then confluence holds.

Many of the confluence lemmata for abstract reduction systems found in literature (see e.g. [8]) are in fact easy corollaries of this theorem. Among the immediate consequences of the theorem are:

(1) the lemma of Hindley–Rosen [5, 10],
(2) Rosen's "requests" lemma [10],
(3) Newman's lemma [9],
(4) Huet's strong confluence lemma [6] and
(5) De Bruijn's lemma [2].

A mediate consequence is the confluence of nonsplitting and relatively terminating reduction systems, a result of Geser [4].

Section 2 contains a short introduction to abstract reduction systems and multisets. In Section 3, we define the lexicographic maximum measure on reductions, and diagrams which are decreasing with respect to this measure. It is shown that these diagrams can be pasted together to form bigger decreasing diagrams. We conclude this section by proving our main theorem. This theorem is applied in Section 4 to obtain the results listed above. The notion of strong confluence is then generalized to abstract reduction systems having more than one reduction relation. We conclude in Section 5 with suggestions for further research.

2. Preliminaries

In this section we give a short introduction to abstract reduction systems and multisets. For an overview of these subjects we refer to [8] and [7].

An abstract reduction system is a set of objects A equipped with some binary "reduction" relations. Throughout this paper, I denotes the set of *labels* (or names) of these relations. Labels will be denoted by α, β and γ.

Definition 2.1 (*Abstract reduction system*). An *Abstract Reduction System* (ARS) is a structure $\mathscr{A} =^{\text{def}} (A, \langle \rightarrow_\alpha \rangle_{\alpha \in I})$ consisting of a set of objects A and a sequence of relations $\rightarrow_\alpha$ on A. A relation $\rightarrow_\alpha$ is said to be a *reduction* relation *labelled* by α. The reduction relation of $\mathscr{A}$ is the union of its constituent reduction relations: $\rightarrow_{\mathscr{A}} =^{\text{def}} \bigcup_{\alpha \in I} \rightarrow_\alpha$. When the ARS is clear from the context, we will suppress it in our notations. Two ARSs $\mathscr{A} =^{\text{def}} (A, \langle \rightarrow_\alpha \rangle_{\alpha \in I})$ and $\mathscr{B} =^{\text{def}} (B, \langle \rightarrow_\beta \rangle_{\beta \in J})$ are *reduction equivalent*, denoted by $\mathscr{A} \leftrightarrows \mathscr{B}$, if $\rightarrow_{\mathscr{A}} = \rightarrow_{\mathscr{B}}$.

Two ARSs which are reduction equivalent can be viewed as different presentations of the same ARS. We extend the introduced notions for relations to ARSs by identifying an ARS $\mathscr{A}$ with its reduction relation $\rightarrow_{\mathscr{A}}$. Such notions obviously do not depend on the presentation of an ARS. For example, λ-calculus (see [1]) can be presented as the ARS $(\Lambda, \langle \beta, \eta \rangle)$, i.e. objects are λ-terms and reduction relations are β- and η-reduction. Another presentation is $(\Lambda, \langle \beta\eta \rangle)$, where $\beta\eta$ is the union of the relations β and η.

We use infix notation for a reduction relation $\rightarrow$ and its derived relations: $\leftarrow, \rightarrow^{=}, \rightarrow^{+}$ and $\twoheadrightarrow$ which denote the inverse, the reflexive closure, the transitive closure and the reflexive–transitive closure of $\rightarrow$, respectively. We use $\rightarrow_\alpha ; \rightarrow_\beta$ to denote the diagrammatic (sequential) composition of $\rightarrow_\alpha$ and $\rightarrow_\beta$. If $a \rightarrow b$, then we speak of a *reduction step* from a to b. An element which cannot be reduced is a *normal form*. A *reduction sequence* or *reduction* is a sequence of reduction steps. The element a is *strongly normalising* (SN) or *terminating*, if all reductions starting with a are finite. The relation $\rightarrow$ is strongly normalising if all elements in its domain are.

The label of a finite reduction is the string of labels of its constituent reduction steps (in the obvious order), i.e. an element of I^*. The symbols σ, τ and υ will be used to denote strings. The concatenation of two strings σ and τ is denoted by $\sigma\tau$.

In this paper we are interested in how the reduction relations of an ARS interact. To visualise this interaction, diagrams are useful and therefore we first fix some diagram notation (which has already been used in the introduction).

Definition 2.2 (*Diagram notation*). A diagram consists of a number of (labelled, dashed) arrows. For diagrams defining a property, the convention will be used that solid arrows are universally quantified and dashed arrows are existentially quantified. It is natural to think of the solid arrows as the *hypothesis* and of the dashed ones as the *conclusion*. By *mirroring* a diagram, we mean mirroring it in its northwest southeast diagonal. The name $\mathscr{N}$ of the property being expressed by or used in a diagram is displayed in its centre. Such a diagram is spoken of as an $\mathscr{N}$-diagram. A property expressed by a diagram whose hypothesis contains only reduction steps is *local*. It is *global* otherwise. Note that if a "double-headed arrow" appears somewhere in the hypothesis of a property, the property is global.

Next we state some commutativity properties using diagrams. Roughly speaking, if two reduction relations commute, then they do not interfere with one another. Properties (1)–(4) are depicted in Fig. 1. Confluence is depicted in the introduction.

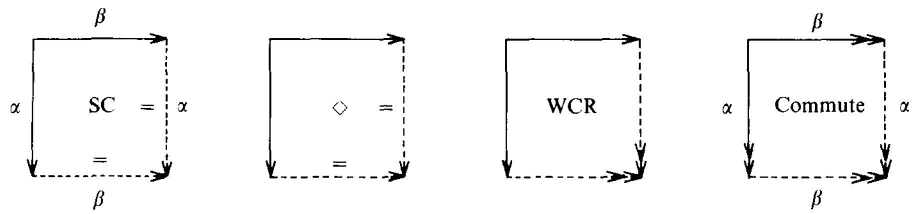


Fig. 1. Commutativity properties.

Definition 2.3 (*Commutativity Properties*). Let $\to_\alpha, \to_\beta$ be two relations on A.

(1) The relation $\to_\alpha$ *subcommutes* (SC) with $\to_\beta$ if $\forall a,b,c\in A$, $\exists d\in A$: $b \leftarrow_\alpha a \to_\beta c \Rightarrow b \to_\beta^{=} d \leftarrow_\alpha^{=} c$, more succinctly expressed by $\leftarrow_\alpha; \to_\beta \subseteq \to_\beta^{=}; \leftarrow_\alpha^{=}$.

(2) A relation is *subcommutative* or has the *diamond* ($\diamond$) property, if it subcommutes with itself.

(3) A relation $\to$ is *locally confluent* or *weakly Church–Rosser* (WCR) if $\leftarrow; \to \subseteq \twoheadrightarrow; \twoheadleftarrow$.

(4) The relations $\to_\alpha$ and $\to_\beta$ *commute* if their transitive–reflexive closures subcommute.

(5) A relation is *confluent* or has the *Church–Rosser* (CR) property if it commutes with itself.

Confluence, the property we are interested in, is an important property in rewriting, because it ensures the uniqueness of the normal form of an object (independent from the question whether such a normal form does exist). It is easy to prove that the local property of subcommutativity implies the global property of confluence. We will show that the subcommutativity requirement can be considerably weakened without loosing confluence. In order for confluence to hold, it is obviously necessary that local confluence holds. The idea is now to choose a presentation of an ARS such that for all local divergences the labels of the convergences needed to reach a common reduct "trace back" to the labels of the local divergence. The labels of reduction sequences are graded by multisets and these will be compared using the multiset extension of the order on the labels of the reduction relations. In this section we assume a fixed strict partial order $\prec$ on the set I of labels of the reduction relations.

Now we give an informal definition of multisets. A formal definition can be found in Appendix A. The definition is slightly more general than the ones usually encountered in literature, to allow for a uniform treatment of both sets and finite multisets. Sets (of labels) will be interpreted as multisets where elements occur either infinitely often or not at all. Usually sets are interpreted as multisets where elements occur at most once. The multiset sum then corresponds to a disjoint union. For the interpretation we have chosen,[1] the multiset sum corresponds to an ordinary union.

[1] The interpretation of sets as infinite multisets was suggested to us by the anonymous referee.

Definition 2.4 (*Multiset*). A (*general*) *multiset* is a collection in which elements are allowed to occur more than once or even infinitely often. The class of multisets over I is denoted by $(M, N, X, Y, Z \in) \mathcal{M}(I)$. A *finite* multiset has finitely many different elements which occur finitely often. The class of finite multisets over I is denoted by $(F, G, H \in) \mathcal{FM}(I)$. A *set* is a multiset in which elements occur either not at all or infinitely often. The class of sets over I is denoted by $(S, T, U \in) \mathcal{SM}(I)$.

Note. In the rest of the paper the type of multiset denoted by a symbol will be as specified above and not be made explicit. For example F, G and H will always denote finite multisets.

When we speak of sets of labels, we always mean the above interpretation of the set as a multiset over the set of labels. To denote operations on multisets we use (the denotations of) the corresponding operations on sets. This causes no confusion, because the operations intersection, union and difference on sets interpreted as multisets have all the usual properties. The multiset sum will be denoted by $\uplus$. For the formal definitions see again Appendix A. Note that both $\mathcal{FM}(I)$ and $\mathcal{SM}(I)$ are closed with respect to intersection, union, sum and difference.

To distinguish between set comprehension and finite multiset comprehension, braces will be used to denote the former and square brackets to denote the latter. For example $[\alpha, \beta]$ denotes the finite multiset with exactly one occurrence of both α and β, whereas $\{\alpha\}$ denotes the set multiset with infinitely many occurrences of α.

The following (in)equalities illustrate the differences between finite and set multisets, and sum and union: $[\alpha] \uplus [\alpha] = [\alpha, \alpha] \neq [\alpha] = [\alpha] \cup [\alpha]$, $\{\alpha\} \uplus \{\alpha\} = \{\alpha, \alpha\} = \{\alpha\} = \{\alpha\} \cup \{\alpha\}$, $[\alpha, \alpha] - [\alpha] = [\alpha]$, $\{\alpha\} - [\alpha, \alpha] = \{\alpha\}$, and $[\alpha, \alpha] - \{\alpha\} = \emptyset$.

The multiset $[\sigma]$ of labels of a string σ is the sum of all label occurrences in it, so in particular we have $[\sigma\tau] = [\sigma] \uplus [\tau]$. For example, if we have digits as labels, $[132343] = [1, 3, 2, 3, 4, 3]$.

The lexicographic maximum measure, to be defined in Definition 3.1, assigns to each reduction a submultiset of the multiset of its label. These multisets will be compared using the standard multiset extension of $\prec$, which inherits well-foundedness (on the class of finite multisets) of $\prec$, as was shown by Dershowitz and Manna [3]. Our definition of the standard multiset extension is a notational variant of the usual Dershowitz–Manna definition. The down-set operator is introduced to allow for algebraic proofs of the properties needed in this paper.

Definition 2.5 (*Multiset extension*). (1) The set $\curlyvee\alpha$ is the strict order ideal generated by (or down-set of) α, defined by $\curlyvee\alpha =^{\text{def}} \{\beta \mid \beta \prec \alpha\}$. This is extended to multisets and strings by defining $\curlyvee M =^{\text{def}} \bigcup_{\alpha \in M} \curlyvee\alpha$ and $\curlyvee\sigma =^{\text{def}} \curlyvee[\sigma]$. For example, $\curlyvee 2 = \curlyvee[0, 2] = \curlyvee 212 = \{0, 1\}$.

(2) The (*standard*) *multiset extension* (denoted by $\prec_{\text{mul}}$) of the partial order $\prec$ is defined by

$$M \prec_{\text{mul}} N \quad \text{if } \exists X, Y, Z\colon M = Z \uplus X, N = Z \uplus Y, X \subseteq \curlyvee Y \text{ and } Y \neq \emptyset.$$

Furthermore, $\preccurlyeq_{\text{mul}}$ will be used to denote the reflexive closure of $\prec_{\text{mul}}$. The relation $\preccurlyeq_{\text{mul}}$ can also be obtained by removing the last condition ($Y \neq \emptyset$) in the definition of $\prec_{\text{mul}}$.

Intuitively, the elements belonging to the down-sets of the multisets are only of minor importance in comparing multisets.[2] Hence we will refer informally to its down-set as "noise generated by" the multiset. Furthermore, we say that M "traces back" to N if $M \preccurlyeq_{\text{mul}} N$. This corresponds to the intuition that each element of M is dominated by some element of N, i.e. traces back to that element. It is well known (cf. [7]) that the standard multiset extension of a strict partial order is again a strict partial order on the class of finite multisets. In the case of general multisets one can show that transitivity is preserved, but irreflexivity is not. For example, $\mathbb{N} <_{\text{mul}} \mathbb{N}$, where $\mathbb{N}$ denotes the (multi)set of natural numbers and $<$ is the natural order on the set of natural numbers.

The following technical lemma paves the way for the confluence theorem in the next section. If one is interested only in the applications of that theorem, it can safely be skipped.

Lemma 2.6 (Properties of the multiset extension).

(1) *Taking the down-set distributes over union and sum.* $\curlyvee(M-N) \supseteq \curlyvee M - \curlyvee N$.

(2) $M \subseteq N \Rightarrow M \preccurlyeq_{\text{mul}} N \Rightarrow \curlyvee M \subseteq \curlyvee N$.

(3) *For finite multisets, we may assume X and Y in Definition* 2.5(2) *to be disjoint.*

(4) *If G is nonempty, then* $F \subseteq \curlyvee G \Rightarrow F \prec_{\text{mul}} G$.

(5) *If* $\curlyvee S \subseteq S$, *then* $F \preccurlyeq_{\text{mul}} G \Leftrightarrow F - S \preccurlyeq_{\text{mul}} G - S$.

(6) *If* $H \subseteq F, G$, *then* $F \preccurlyeq_{\text{mul}} G \Leftrightarrow F - H \preccurlyeq_{\text{mul}} G - H$.

(7) *If* $H \subseteq \curlyvee G - \curlyvee F$, *then* $F \preccurlyeq_{\text{mul}} G \Leftrightarrow F \uplus H \preccurlyeq_{\text{mul}} G$.

Proof. (1) By definition, noting that union coincides with sum for sets. For the inequality to hold, it suffices to note that $\curlyvee M - \curlyvee N \subseteq \bigcup_{\alpha \in M, \nexists \beta \in N, \alpha \preccurlyeq \beta} \curlyvee \alpha \subseteq \bigcup_{\alpha \in M-N} \curlyvee \alpha = \curlyvee(M-N)$.

(2) Easy.

(3) If they are not disjoint, we can take $Z' =^{\text{def}} Z \uplus (X \cap Y)$, $X' =^{\text{def}} X - Y$, and $Y' =^{\text{def}} Y - X$. By Lemma A.3(10) we have $M = Z' \uplus X'$, $N = Z' \uplus Y'$ and X' and Y'

[2] The correspondence between the intuition and the formal definition of the multiset extension is not exact. It looks like coincidence that noise is not important in comparing multisets using the multiset extension. We obtain a better match if we define the order extension $\prec_{\text{m}}$ of $\prec$ by

$$M \prec_{\text{m}} N =^{\text{def}} \partial M \prec_{\text{mul}} \partial N \quad \text{or} \quad (\partial M = \partial N \;\&\; \mathring{M} \prec_{\text{m}} \mathring{N})$$

where $\partial M =^{\text{def}} M - \curlyvee M$, the *boundary* (i.e. maximal elements) of M and $\mathring{M} =^{\text{def}} M \cap \curlyvee M$, the *interior* (i.e. noise) of M. In words this reads, first compare the maximal elements of the multisets and only if this is not decisive apply the method recursively to its noise. This order is the same as the order $\ll_{\cdot \mathcal{M}}$ in [7], where it is shown that it properly contains the standard multiset extension and inherits well-foundedness. Although the properties needed in this paper hold for both extensions, we will prove this only for the standard multiset extension.

are obviously disjoint. Furthermore, $X' \subset \curlyvee Y'$ holds because $X - Y \subset \curlyvee Y$ by the assumption $X \subset \curlyvee Y$, $\curlyvee Y = \curlyvee(Y - \curlyvee Y)$ because in finite multisets there are no infinite ascending chains, and finally $\curlyvee(Y - \curlyvee Y) \subseteq \curlyvee(Y - X)$ again using the assumption.

(4) By definition.

(5) The implication from right to left is a direct consequence of the definition, so the other direction remains to be shown. By definition there exist X, Y and Z such that $F = Z \uplus X$, $G = Z \uplus Y$ and $X \subseteq \curlyvee Y$. Because set-difference distributes over sum (Lemma A.3(8)) we only have to show $X - S \subseteq \curlyvee(Y - S)$. This is immediate from (1), $\curlyvee S \subseteq S$, and the assumption $X \subseteq \curlyvee Y$.

(6) The implication from right to left is again trivial. The other direction is more difficult to show. By definition there exist X, Y and Z such that $F = Z \uplus X$, $G = Z \uplus Y$ and $X \subseteq \curlyvee Y$, and we may assume by (3) that X and Y are disjoint. From this and the assumption $H \subseteq F, G$, we conclude that $H \subseteq Z$. Subtracting elements from Z has no influence on the order.

(7) The if-direction is a consequence of (2). For the only-if-direction, we can take X, Y and Z as in the proof of (5). By the assumption $H \subseteq \curlyvee G - \curlyvee F$ and (1) it holds that $H \subseteq \curlyvee Y$, so $X \uplus H \subseteq \curlyvee Y$ and we are done. □

Note. All the statements in the above lemma remain true if we replace the occurrences of F and G by M and N. Since finite multisets suffice for our purposes, we do not prove this. Actually, one easily verifies that the only proof that has to be modified is the proof of (6). This is necessary, because (3) cannot be extended to general multisets, as exemplified by $[0, 1, 2, \ldots] <_{\mathrm{mul}} [1, 2, \ldots]$.

3. Confluence by decreasing diagrams

In this section we prove a general theorem for deriving confluence from local confluence. We do this by gluing together small "decreasing" tiles into bigger ones having that same property. The diagrams are decreasing in the sense that their conclusion is less than or equal to their hypothesis. First we define a measure on strings of labels and hence on reduction sequences labelled by them. In this section we assume the set of labels I to be strictly partially ordered by $\prec$.

Definition 3.1 (*Lexicographic maximum measure*). The (*lexicographic maximum*) *measure* is a map $|.| : I^* \to \mathscr{FM}(I)$, grading strings by finite multisets. It is inductively defined as follows.

- $|\varepsilon| =^{\mathrm{def}} [\,]$,
- $|\alpha\sigma| =^{\mathrm{def}} [\alpha] \uplus (|\sigma| - \curlyvee\alpha)$.

For example $|132343| = [1, 3, 3, 4]$ and $|211| = [2]$. Intuitively, we take the multiset of elements which are maximal (in the $\prec$ ordering) with respect to the elements to their left in the string. Operationally, one can think of filtering out the noise before proceeding to the right. The measure of a reduction is the measure of its label. For example, $|a \to_2 b \to_1 c| = |21| = [2]$. The measure of a diagram is the multiset sum of the measures of the reductions in its hypothesis.

The next lemma shows how the measure of a string can be decomposed into the sum of the measures of its substrings.

Lemma 3.2 (Properties of lexicographic maximum measure).
(1) $\curlyvee|\sigma| = \curlyvee\sigma$.
(2) $|\sigma\tau| = |\sigma| \uplus (|\tau| - \curlyvee\sigma)$.

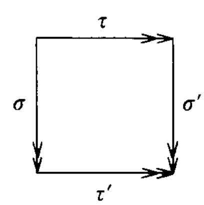
Proof. (1) The sets of maxima of $|\sigma|$ and $[\sigma]$ are easily shown to be the same. In the sequel we will make use of this fact without mention.

(2) By induction on the length of σ. The base case being trivial, we only show the induction step.

$$
\begin{aligned}
|\alpha\sigma\tau| &= [\alpha] \uplus (|\sigma\tau| - \curlyvee\alpha) && \text{(definition)}\\
&= [\alpha] \uplus ((|\sigma| \uplus (|\tau| - \curlyvee\sigma)) - \curlyvee\alpha) && \text{(induction hypothesis)}\\
&= [\alpha] \uplus (|\sigma| - \curlyvee\alpha) \uplus ((|\tau| - \curlyvee\sigma) - \curlyvee\alpha) && \text{(distribute (A.3(8)))}\\
&= |\alpha\sigma| \uplus ((|\tau| - \curlyvee\sigma) - \curlyvee\alpha) && \text{(definition)}\\
&= |\alpha\sigma| \uplus (|\tau| - \curlyvee\alpha\sigma) && \text{(Lemmas A.3(9) and 2.6(1))} \quad \square
\end{aligned}
$$

The lexicographic maximum measure is designed to make pasting decreasingness preserving (Lemma 3.5) and hypothesis decreasing (Lemma 3.6). The intuition for this measure is that labels below a label in front of them do not matter in proving confluence. At the time we get to them (to finish the confluence diagram) we already know they "behave nicely" because the bigger label does so. Now we define decreasing diagrams with respect to this measure.

Definition 3.3 (*Decreasing diagram*). The diagram



is *decreasing* (D) if the following decreasingness[3] condition is satisfied:

$$|\sigma\tau'| \preccurlyeq_{\text{mul}} |\tau| \uplus |\sigma| \succcurlyeq_{\text{mul}} |\tau\sigma'|$$

[3] We can extend the notion of decreasingness a little bit by taking $\preccurlyeq_{\text{m}}$ instead of $\preccurlyeq_{\text{mul}}$ for comparing the measures. However, these notions coincide for the case of a locally decreasing diagram.

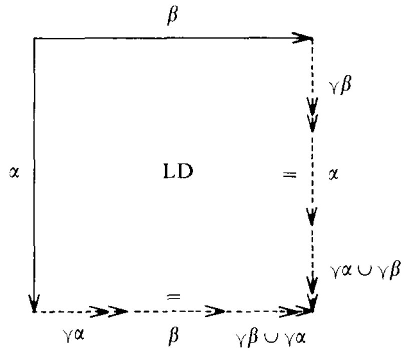


Fig. 2. Locally decreasing diagram.

A *locally decreasing* (LD) diagram is a decreasing diagram such that its divergence is local, i.e. both σ and τ consist of exactly one step.

Using the Decomposition Lemma 3.2(2) and the property in Lemma 2.6(6) we can reformulate the decreasingness condition as

$$|\tau'| - \curlyvee\sigma \preccurlyeq_{\mathrm{mul}} |\tau| \quad \text{and} \quad |\sigma| \succcurlyeq_{\mathrm{mul}} |\sigma'| - \curlyvee\tau.$$

One can think of these inequalities in the following way. The labels in the measure of the conclusion (σ') all trace back to the labels (in the measure) of the opposite side in the hypothesis (σ), except for the noise (elements of $\curlyvee\tau$) which has been generated by the adjacent side in the hypothesis (τ).

The next proposition gives a characterisation of the convergent reduction sequences of a locally decreasing diagram.

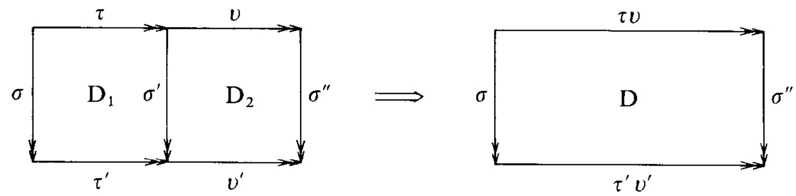
Proposition 3.4. *The form of a locally decreasing diagram is as specified in Fig.* 2.

Proof. The reformulation of decreasingness in the case of a local diagram where $\sigma = \alpha$ and $\tau = \beta$, reads $|\tau'| - \curlyvee\alpha \preccurlyeq_{\mathrm{mul}} [\beta]$ and $[\alpha] \succcurlyeq_{\mathrm{mul}} |\sigma'| - \curlyvee\beta$. It is apparent that σ' is a string less than or equal to α, i.e. a string headed by at most one α followed by a number of labels less than α, interspersed with noise from β, i.e. labels less than β. This is described exactly by the right-hand side of the LD-diagram.[4] The same holds, mutatis mutandis, for τ'. □

A nice property of D-diagrams is that they can be pasted together to form bigger D-diagrams.

Lemma 3.5 (Pasting preserves decreasingness).

[4] If one replaces $\preccurlyeq_{\mathrm{mul}}$ by $\preccurlyeq_{\mathrm{m}}$, then nothing changes because one easily proves $F \preccurlyeq_{\mathrm{mul}} [\alpha] \Leftrightarrow F \preccurlyeq_{\mathrm{m}} [\alpha]$, for every finite multiset F.



Proof. We have to prove that the diagram on the right is decreasing. Continuing on the informal explanation of decreasingness, the proofs are guided by tracing back the reductions in the conclusion to the reductions in the hypothesis. For the right-hand side of the conclusion, the labels of σ'' are either noise from υ or trace back to σ'. The labels of σ' are either noise from τ or trace back to σ. We can combine these observations by noting that all the noise generated by τ and υ can also be generated by $\tau\upsilon$, and that tracing back is transitive. Formally

$$\begin{aligned}
|\tau\upsilon\sigma''| &= |\tau\upsilon| \uplus (|\sigma''| - \Upsilon\tau\upsilon) && \text{(decompose (3.2(2)))}\\
&= |\tau\upsilon| \uplus ((|\sigma''| - \Upsilon\upsilon) - \Upsilon\tau) && \text{(Lemmas A.3(9) and 2.6(1))}\\
&\preccurlyeq_{\mathrm{mul}} |\tau\upsilon| \uplus (|\sigma'| - \Upsilon\tau) && (\mathrm{D}_2)\\
&\preccurlyeq_{\mathrm{mul}} |\tau\upsilon| \uplus |\sigma| && (\mathrm{D}_1).
\end{aligned}$$

Observe the close relationship between the informal and formal proofs. For the other side of the conclusion the situation is more complicated. For the first half of the conclusion (τ'), everything is straightforward. However, for the second half of the conclusion (υ'), the noise generated by σ' either traces back to σ or it is the noise generated by τ. The former case is not problematic, because σ is allowed to generate noise inside $\tau'\upsilon'$. The latter case is problematic, because it is not clear why steps in υ' should trace back to τ. We are saved by the lexicographic maximum measure because, roughly speaking, some of the steps in υ' are filtered out by τ' in taking the measure of $\tau'\upsilon'$ and the other ones can safely be traced back to τ (safely, because they were *not* filtered out):

$$\begin{aligned}
|\sigma\tau'\upsilon'| &= |\sigma\tau'| \uplus (|\upsilon'| - \Upsilon\sigma\tau') && \text{(decompose (3.2(2)))}\\
&= |\sigma\tau'| \uplus ((|\upsilon'| - \Upsilon\sigma\tau') \cap \Upsilon\tau) \uplus ((|\upsilon'| - \Upsilon\sigma\tau') - \Upsilon\tau) && \text{(split (A.3(10)))}\\
&\preccurlyeq_{\mathrm{mul}} |\sigma| \uplus |\tau| \uplus ((|\upsilon'| - \Upsilon\sigma\tau') - \Upsilon\tau) && \text{(see below)}\\
&\subseteq |\sigma| \uplus |\tau| \uplus ((|\upsilon'| - \Upsilon\sigma') - \Upsilon\tau) && \text{(see below)}\\
&\preccurlyeq_{\mathrm{mul}} |\sigma| \uplus |\tau| \uplus (|\upsilon| - \Upsilon\tau) && (\mathrm{D}_2)\\
&= |\sigma| \uplus |\tau\upsilon| && \text{(compose (3.2(2))).}
\end{aligned}$$

The correspondence between the informal proof and the formal one is here more difficult to find, but still present. For example, the first not yet justified step corresponds exactly to the problematic case above.

Claim. *If we take* $F =^{\text{def}} |\sigma\tau'|$, $G =^{\text{def}} |\sigma| \uplus |\tau|$, *and* $H =^{\text{def}} (|\upsilon'| - \curlyvee\sigma\tau') \cap \curlyvee\tau$, *then the assumptions of Lemma* 2.6(7) *are satisfied. Hence the step is justified.*

Proof of claim. The first assumption, $F \preccurlyeq_{\text{mul}} G$, follows directly from D_1. The second one, $H \subseteq \curlyvee G - \curlyvee F$, is shown by the following simple argument:

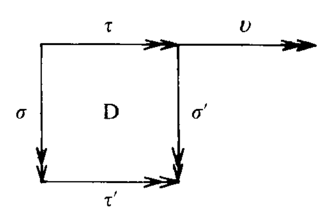
$$\begin{aligned}
(|\upsilon'| - \curlyvee\sigma\tau') \cap \curlyvee\tau &= (|\upsilon'| \cap \curlyvee\tau) - \curlyvee\sigma\tau' && \text{(exchange (A.3(11)))}\\
&\subseteq \curlyvee\tau - \curlyvee\sigma\tau' \\
&\subseteq (\curlyvee\sigma \uplus \curlyvee\tau) - \curlyvee\sigma\tau' \\
&= \curlyvee(|\sigma| \uplus |\tau|) - \curlyvee|\sigma\tau'| && \text{(distribute (2.6(1)) and Lemma 3.2(1)).}
\end{aligned}$$

This proves the claim. The other step reformulates which steps in υ' need to trace back to υ. This is just as simple:

$$\begin{aligned}
(|\upsilon'| - \curlyvee\sigma\tau') - \curlyvee\tau &= |\upsilon'| - \curlyvee\sigma\tau'\tau && \text{(Lemma A.3(9))}\\
&\subseteq |\upsilon'| - \curlyvee\sigma\tau \\
&\subseteq |\upsilon'| - \curlyvee(|\sigma'| \uplus |\tau|) && (D_1)\\
&= (|\upsilon'| - \curlyvee\sigma') - \curlyvee\tau && \text{(Lemma A.3(9)).} \qquad \square
\end{aligned}$$

We will prove the main theorem by well-founded induction on the measure of a diagram. The next lemma states that by filling in a decreasing diagram, the measure is decreased.

Lemma 3.6 (Pasting is hypothesis decreasing). *If* τ *is nonempty and we have the situation*



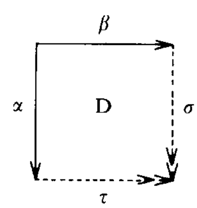
then $|\sigma'| \uplus |\upsilon| \prec_{\text{mul}} |\sigma| \uplus |\tau\upsilon|$, *i.e. the measure of the hypothesis is decreased.*

Proof. What labels are in the measure of the new hypothesis? A label in the measure of σ' either traces back to σ or is noise generated by τ. The labels in the measure of υ either were also present in the measure of $\tau\upsilon$ or were filtered out by τ. In the last case they can be considered as noise generated by τ. Summing up, the only "created" step labels in the new hypothesis can be seen as noise generated by τ:

$$\begin{aligned}
|\sigma'|\uplus|\upsilon| &= ((|\sigma'|\uplus|\upsilon|)\cap\curlyvee\tau)\uplus((|\sigma'|\uplus|\upsilon|)-\curlyvee\tau) && \text{(split (A.3(10)))}\\
&\prec_{\text{mul}} |\tau|\uplus((|\sigma'|\uplus|\upsilon|)-\curlyvee\tau) && \text{(noise reduction (2.6(4)))}\\
&= |\tau|\uplus(|\sigma'|-\curlyvee\tau)\uplus(|\upsilon|-\curlyvee\tau) && \text{(distribute (A.3(8)))}\\
&= |\tau\sigma'|\uplus(|\upsilon|-\curlyvee\tau) && \text{(compose (3.2(2)))}\\
&\preccurlyeq_{\text{mul}} |\sigma|\uplus|\tau|\uplus(|\upsilon|-\curlyvee\tau) && \text{(D)}\\
&= |\sigma|\uplus|\tau\upsilon| && \text{(compose (3.2(2)).} \quad \square
\end{aligned}$$

Now the two previous lemmas can be used in a straightforward way to obtain our main theorem. It is only here that we have to assume that the strict partial order $\prec$ is well-founded.

Main Theorem 3.7. *Let* $\mathscr{A} =^{\text{def}} (A, \langle\rightarrow_\alpha\rangle_{\alpha\in I})$ *be an ARS and let* $\prec$ *be a well-founded partial order on* I. *Let* I_{v} *and* I_{h} *be (not necessarily disjoint) subsets of* I, *with* $\rightarrow_{\text{v}} =^{\text{def}} \bigcup_{\alpha\in I_{\text{v}}}\rightarrow_\alpha$ *and* $\rightarrow_{\text{h}} =^{\text{def}} \bigcup_{\beta\in I_{\text{h}}}\rightarrow_\beta$. The v *and* h *stand for vertical and horizontal. If, for all* α *in* I_{v} *and* β *in* I_{h}, *the following diagram holds, then* $\rightarrow_{\text{v}}$ *commutes with* $\rightarrow_{\text{h}}$:



In the diagram σ *must consist of vertical labels* ($\in I_{\text{v}}$) *and* τ *of horizontal ones* ($\in I_{\text{h}}$).

Proof. The theorem is proved by well-founded induction on the measure of diagrams, showing that we always obtain D-diagrams. The proof is expressed by the diagram in Fig. 3.

The diagram IH_1 can be completed to a D-diagram because of the induction hypothesis, using Lemma 3.6. The diagrams LD and IH_1 together form a D-diagram by Lemma 3.5, hence the diagram IH_2 can be completed again to a D-diagram because of the induction hypothesis, using (the mirrored version of) Lemma 3.6. Now the complete diagram is a D-diagram by another appeal to (the mirrored version of) Lemma 3.5. $\square$

Note. The above proof can be made formal, by showing that the predictate P on divergences, defined by: $P(b \twoheadleftarrow a \twoheadrightarrow c) =^{\text{def}}$ *$\exists d$, $b \twoheadrightarrow d \twoheadleftarrow c$ forming a decreasing diagram*, is a $\succ_{\text{mul}}$-complete[5] predicate (see [6]).

[5] For the extended notion of decreasingness one must show P to be $\succ_{\text{m}}$-complete.

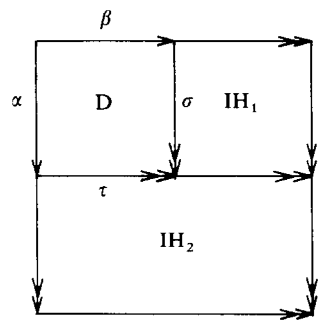


Fig. 3. Proof of main theorem.

A special case of the theorem arises when we take the sets I_v and I_h to be equal to the set of all labels I. In searching for applications of the theorem, the characterization of locally decreasing diagrams (Proposition 3.4) is often helpful. Another useful observation is that for proving confluence the presentation of an ARS can be chosen freely.

Definition 3.8 (*LD and DCR*). (1) An ARS $\mathscr{A} =^{\text{def}} (A, \langle \rightarrow_\alpha \rangle_{\alpha \in I})$ is *locally decreasing* (LD), if there exist a relation $\prec$ and sets I_v and I_h satisfying the assumptions of Theorem 3.7, such that $\rightarrow_{\mathscr{A}} = \rightarrow_v = \rightarrow_h$.

(2) An ARS is *decreasing Church–Rosser* (DCR) if it is reduction equivalent to a locally decreasing ARS.

In the sequel, if we do not specify the sets I_v and I_h, they are assumed to be equal to I, the set of all labels.

Corollary 3.9. (1) *A locally decreasing ARS is confluent.*

(2) *A decreasing Church–Rosser ARS is confluent.*

Proof. (1) By assumption we can apply Theorem 3.7 and conclude that $\rightarrow_{\mathscr{A}}$ commutes with itself, that is, $\mathscr{A}$ is confluent.

(2) Directly by (1), noting that the presentation of an ARS does not influence confluence. □

4. Applications

In this section we apply the results from the previous section to obtain proofs of some classical confluence lemmata. The difficulty in applying our main theorem to an ARS will be finding a suitable presentation of the ARS and finding a well-founded partial order on its set of labels, such that it is locally decreasing. The first application will be the lemma of Hindley–Rosen. As the ordering on the labels models how the reduction steps interact and in the lemma of Hindley–Rosen there is no interaction at all, the application is straightforward.

Corollary 4.1 (Lemma of Hindley–Rosen [5, 10]). *Let* $(A, \langle \rightarrow_\alpha \rangle_{\alpha \in I})$ *be an ARS. If for all* α *and* β *in* I, $\rightarrow_\alpha$ *subcommutes with* $\rightarrow_\beta$, *then* $\mathscr{A}$ *is confluent.*

Proof. Take for $\prec$ the empty order on I. Because an SC-diagram is a special LD-diagram (compare Figs. 1 and 2), we can apply Corollary 3.9(1) and obtain confluence of $\mathscr{A}$. □

For the next application the ordering is almost as easy to find.

Definition 4.2 (*Requests* [10]). Let $\rightarrow_1$ and $\rightarrow_2$ be relations on A. The relation $\rightarrow_2$ *requests* $\rightarrow_1$ if $\leftarrow_1;\rightarrow_2 \subseteq \rightarrow_2;\ \rightarrow_1;\leftarrow_1$ (see Fig. 4). Informally, the second reduction relation requests the first one to reach a common reduct.

Corollary 4.3 (Requests lemma [10]). *Let* $(A, \langle \rightarrow_1, \rightarrow_2 \rangle)$ *be an ARS. If* $\rightarrow_1$ *and* $\rightarrow_2$ *both are subcommutative and* $\rightarrow_2$ *requests* $\rightarrow_1$, *then* $\mathscr{A}$ *is confluent.*

Proof. Take as order $1 \prec 2$. Now $\mathscr{A}$ is confluent because SC-diagrams as well as requests-diagrams and mirrored requests-diagrams are LD-diagrams, so Corollary 3.9(1) can be applied again. □

Note. Actually, the results above are trivial reformulations of the original ones by Hindley and Rosen.

As a simple example of an application of the main theorem to a specific ARS, we show that β- and η-reductions commute for λ-calculus. By simple case analysis (see [1]) one shows that the diagram in Fig. 4 holds. Now one notes that β has "splitting effect" on η but not vice versa, so if we take $\eta \prec \beta$, then the diagram is decreasing hence β commutes with η. The same method cannot be applied to obtain that β is confluent, the main difficulty being that β-reduction has splitting effect on itself.

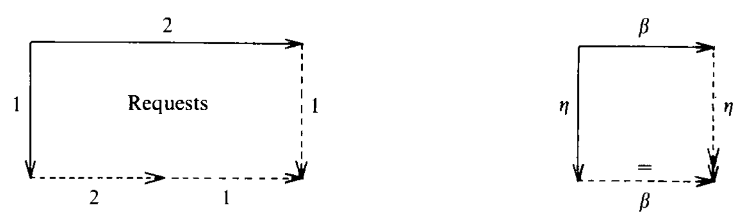


Fig. 4. Requests and λ-calculus.

The results so far could be obtained by constructing a well-founded ordering on the set of labels of the reduction relations, i.e. showing the ARSs to be locally decreasing. For the remaining applications, we shall have to construct suitable presentations of the ARSs as well. The idea is that we have to choose a presentation such that reduction steps have no splitting effect on themselves. If the reduction relation is strongly normalising then the splitting behaviour does not matter, as stated by an early result of Newman.

Corollary 4.4 (Newman's lemma [9]). *Let* $\mathscr{A} =^{\text{def}} (A, \to)$ *be an ARS. If* $\mathscr{A}$ *is locally confluent and strongly normalising, then* $\mathscr{A}$ *is confluent.*

Proof. Let the reduction relations of the ARS $\mathscr{B} =^{\text{def}} (A, \langle \to_a \rangle_{a \in A})$ be defined as follows: $a \to_a b$ if $a \to b$, for all a, b in A. Let $\prec =^{\text{def}} \leftarrow^+$, then $\prec$ is a well-founded partial order because $\to$ is strongly normalising. Obviously, we have $\mathscr{A} \leftrightarrows \mathscr{B}$. The translation of a WCR-diagram in $\mathscr{A}$ to a diagram in $\mathscr{B}$ is shown in Fig. 5. Note that a reduction sequence $a \to b \twoheadrightarrow d$ in $\mathscr{A}$ translates to a reduction sequence $a \to_a b \twoheadrightarrow_\sigma d$ in $\mathscr{B}$ such that a is greater than every element in σ. Thus, the simulation by $\mathscr{B}$ of a WCR-diagram in $\mathscr{A}$ is an LD-diagram. We conclude from Corollary 3.9(2) that $\mathscr{A}$ is DCR, hence CR. $\square$

If one can choose the splitting to take place in one direction only, as in Huet's Lemma below, then confluence can also be proved.

Definition 4.5 (*Strong confluence* [6]). A relation $\to$ is *strongly* confluent (SCR) if $\leftarrow ; \to \subseteq \to^{=} ; \twoheadleftarrow$ (see Fig. 6). Informally, the tiler can choose the side of the splitting.

Although a direct proof that strong confluence implies confluence is easy, we want to reduce this problem to Theorem 3.7. It is not immediately clear how this can be done, because the SCR-diagram does not fit in the LD-diagram. This is because a reduction step is split into several reduction steps having the same label, not smaller ones as required by the theorem. The solution is to note that splitting takes place only

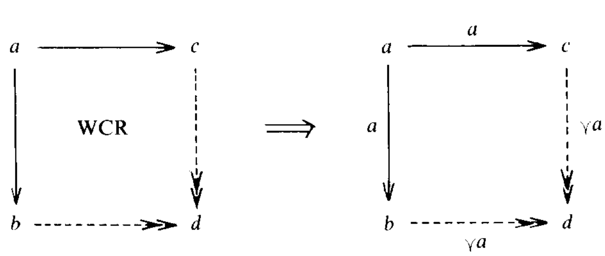


Fig. 5. Proof of Newman's lemma.

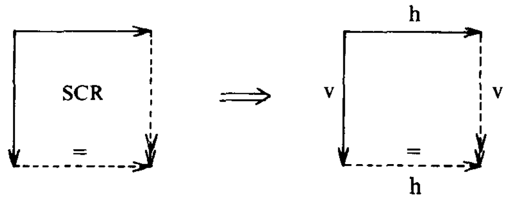


Fig. 6. Proof of Huet's lemma.

in the vertical reductions. Hence, we make a distinction between vertical and horizontal reductions, ordering the horizontal steps above the vertical ones.

Corollary 4.6 (Strong confluence lemma [6]). *Let $\mathscr{A} =^{\mathrm{def}} (A,\rightarrow)$ be an ARS. If $\mathscr{A}$ is strongly confluent, then $\mathscr{A}$ is confluent.*

Proof. Define the ARS $\mathscr{B} =^{\mathrm{def}} (A, \langle \rightarrow_{\mathrm{h}}, \rightarrow_{\mathrm{v}} \rangle)$ by $\rightarrow_{\mathrm{h}} =^{\mathrm{def}} \rightarrow_{\mathrm{v}} =^{\mathrm{def}} \rightarrow$. Obviously, $\mathscr{A} \rightleftarrows \mathscr{B}$. The idea is to simulate the "vertical" reduction steps in the SCR-diagram of $\mathscr{A}$ by $\rightarrow_{\mathrm{v}}$ and the "horizontal" reduction steps by $\rightarrow_{\mathrm{h}}$ in $\mathscr{B}$. This translation is depicted in Fig. 6. By an appeal to Theorem 3.7 (taking as order $\mathrm{v} \prec \mathrm{h}$)$\rightarrow_{\mathrm{h}}$ commutes with $\rightarrow_{\mathrm{v}}$. From this we conclude immediately by Corollary 3.9(2) that $\mathscr{A}$ is confluent. □

The method of proof of the previous lemma can be extended easily to ARSs having more than one reduction relation. This leads to an extended notion of strong confluence, coinciding with the usual one in the case of an ARS with one reduction relation. We obtain an asymmetrical version of the main theorem.

Definition 4.7 (*Strong confluence (Extended)*). Let $\mathscr{A} =^{\mathrm{def}} (A, \langle \rightarrow_\alpha \rangle_{\alpha \in I})$ be an ARS. Let $\prec$ be a well-founded partial order on I. The ARS $\mathscr{A}$ is *strongly confluent* if for all α and β in I, we have the diagram in Fig. 7 or (cf. Proposition 3.4)

$$|\tau'| - \curlyvee\alpha \preccurlyeq_{\mathrm{mul}} [\beta] \quad \text{and} \quad [\alpha] \succcurlyeq_{\mathrm{mul}} [\sigma'] - {\downarrow}\beta$$

where τ' is the bottom side, σ' the right-hand side of the conclusion, and ${\downarrow}\beta$ denotes the order ideal generated by β, defined by ${\downarrow}\beta =^{\mathrm{def}} \curlyvee\beta \cup \{\beta\}$.

Theorem 4.8. *Every strongly confluent ARS is confluent.*

Proof. Let $\prec$ be the well-founded partial order on I making the ARS $\mathscr{A} =^{\mathrm{def}} (A, \langle \rightarrow_\alpha \rangle_{\alpha \in I})$ strongly confluent. We adapt the method used in the proof of Corollary 4.6. We create for every reduction relation a horizontal and a vertical version. Let the ARS $\mathscr{B} =^{\mathrm{def}} (A, \langle \rightarrow_\alpha \rangle_{\alpha \in I'})$ be defined by $I' =^{\mathrm{def}} I_{\mathrm{h}} \cup I_{\mathrm{v}}$, where $I_x =^{\mathrm{def}} \{\alpha_x \mid \alpha \in I\}$ and $\rightarrow_{\alpha_x} =^{\mathrm{def}} \rightarrow_\alpha$, for x in $\{\mathrm{h}, \mathrm{v}\}$, then we have $\mathscr{A} \leftrightarrows \mathscr{B}$. Define the order $\prec'$ on I' by $\alpha_x \prec' \beta_y$ if either $\alpha \prec \beta$ or $\alpha = \beta$, $x = \mathrm{v}$ and $y = \mathrm{h}$. It is easy to check that

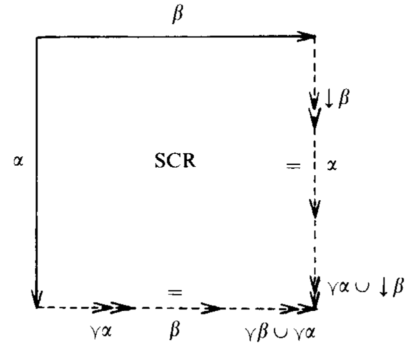
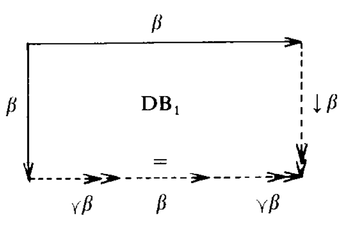


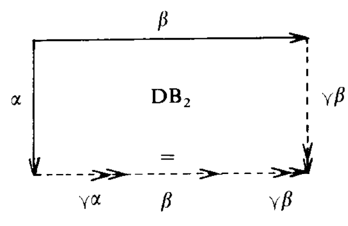
Fig. 7. Extended strong confluence.

$\prec'$ is a well-founded partial order on I'. We translate an SCR-diagram in $\mathscr{A}$ to an SCR-diagram in $\mathscr{B}$ by simulating vertical reduction steps by $\rightarrow_v$ and horizontal ones by $\rightarrow_h$. Now in order to prove that $\rightarrow_v$ commutes with $\rightarrow_h$, we must check that (the translation of) an SCR-diagram is an LD-diagram. Comparing Figs. 2 and 7, this is easily seen to be the case, because the only steps which might cause problems, the vertical β-steps, are less than the horizontal β-steps with respect to $\prec'$. This shows that $\mathscr{B}$ is locally decreasing, $\mathscr{A}$ is decreasing Church–Rosser, and by Corollary 3.9(2) that $\mathscr{A}$ is confluent. □

We next state two corollaries of this theorem. The first one is a lemma by De Bruijn [2]. It was the search for a simple proof of this lemma, instead of the complicated combinatorial proof given in his paper, which led to our notion of decreasing diagram.

Corollary 4.9 (De Bruijn's Lemma [2]). *Let $\mathscr{A} =^{\text{def}} (A, \langle \rightarrow_\alpha \rangle_{\alpha \in I})$ be an ARS. Let $\prec$ be a well-founded total order on I. If for all $\alpha \prec \beta$ in I, we have the following diagrams, then $\mathscr{A}$ is confluent.*





Proof. A well-founded total order is of course a well-founded partial order. One easily checks that the diagrams DB_1, DB_2 and the mirrored version of DB_2 all are SCR-diagrams, so we can apply Theorem 4.8 and obtain confluence. Note that because of the totality of the order $\prec$, these three cases cover all the possible local divergences. The mirrored version of DB_2 is needed because of the condition $\alpha \prec \beta$ on its hypothesis. □

Next, we show a nontrivial application of the strong confluence theorem to obtain a recent result by Geser.

Definition 4.10 (*Relative termination and non-splitting* [4]). Let $\mathscr{A} =^{\mathrm{def}} (A, \langle \rightarrow_\alpha, \rightarrow_\beta \rangle)$ be an ARS.

(1) The relation $\rightarrow_\alpha$ *modulo* $\rightarrow_\beta$ is defined by $\rightarrow_{\alpha/\beta} =^{\mathrm{def}} \twoheadrightarrow_\beta; \rightarrow_\alpha; \twoheadrightarrow_\beta$. If $\rightarrow_{\alpha/\beta}$ is terminating, then $\rightarrow_\alpha$ is said to be *relatively terminating* (with respect to $\rightarrow_\beta$).

(2) The relation $\rightarrow_\beta$, is *nonsplitting* (NS), if $\leftarrow_\beta; \rightarrow_{\mathscr{A}} \subseteq \rightarrow_{\mathscr{A}}^{=}; \twoheadleftarrow_{\mathscr{A}}$ (see Figs. 8 and 9).

Corollary 4.11 [4]. *Let $\mathscr{A} =^{\mathrm{def}} (A, \langle \rightarrow_\alpha, \rightarrow_\beta \rangle)$ be an ARS. If $\rightarrow_\alpha$ is relatively terminating and locally confluent and if $\rightarrow_\beta$ is nonsplitting, then $\mathscr{A}$ is confluent.*

Proof. The idea of proof is the same as for the proof of Newman's lemma (Corollary 4.4). The reduction relations are split into smaller ones based on the "weight" (with respect to $\rightarrow_{\alpha/\beta}$) of the origin of a reduction step. Let the reduction relations of the ARS $\mathscr{B} =^{\mathrm{def}} (A, \langle \rightarrow_a \rangle_{a \in A})$ be defined as follows: $a \rightarrow_c b$ if $c \twoheadrightarrow_\beta a \rightarrow_{\mathscr{A}} b$, for all a, b and c in A. Note that we can translate a reduction step in $\mathscr{A}$ to many different reduction steps in $\mathscr{B}$, but nevertheless $\mathscr{A}$ and $\mathscr{B}$ are clearly reduction equivalent. Let $\prec =^{\mathrm{def}} \overset{+}{\leftarrow}_{\alpha/\beta}$, then $\prec$ is a well-founded partial order because $\rightarrow_\alpha$ is relatively terminating. To obtain confluence of $\mathscr{A}$ it suffices to check that $\mathscr{B}$ is strongly confluent. So

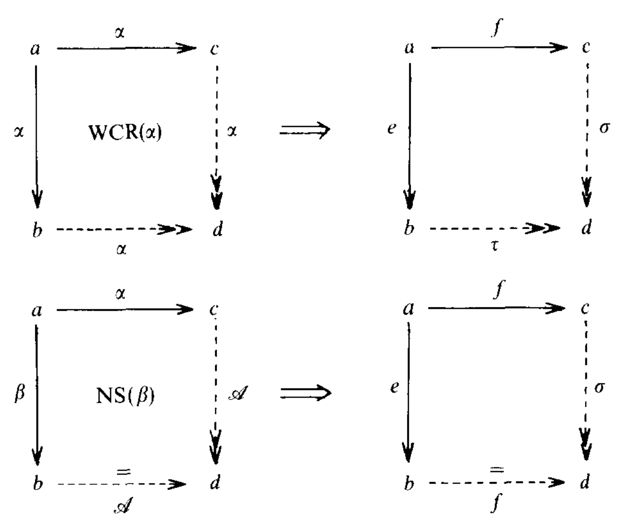


Fig. 8. Proof of Geser's lemma.

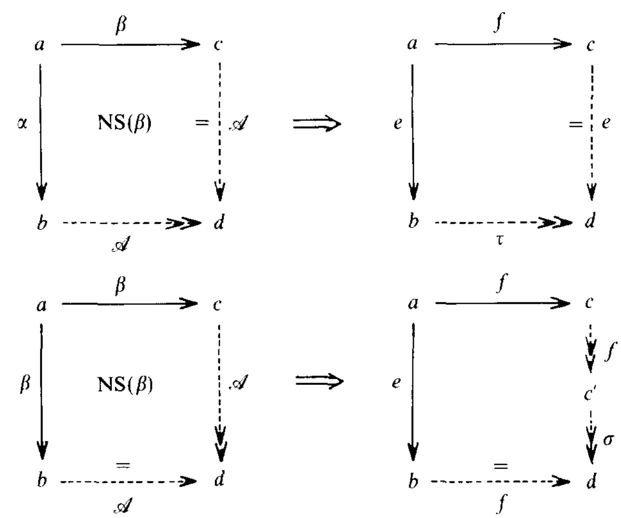


Fig. 9. Proof of Geser's lemma.

suppose we have a local divergence $b \leftarrow_e a \rightarrow_f c$ in $\mathscr{B}$. Then by definition there exist reductions $e \twoheadrightarrow_\beta a \rightarrow_\mathscr{A} b$ and $f \twoheadrightarrow_\beta a \rightarrow_\mathscr{A} c$ in $\mathscr{A}$. Because in both of these reductions the $\rightarrow_\mathscr{A}$-step can be either an $\rightarrow_\alpha$- or a $\rightarrow_\beta$-step, there are four cases to consider. These cases are depicted in Figs. 8 and 9. The e- and f-labels in the conclusions are justified because, e.g. $f \twoheadrightarrow_\beta a \rightarrow_\beta b \rightarrow_\mathscr{A} d$. The labels of σ and τ in the conclusions are obtained by translating the corresponding reductions on the left. Each step $a' \rightarrow b'$ is translated to $a' \rightarrow_{a'} b'$. By choice of the order all the labels in σ and τ are then below f and e, respectively. The only problem occurs in the last case of Fig. 9. There the labels in the right-hand side of the conclusion cannot be chosen below f until the first (if any) α-step occurs. Happily, we can choose them to be equal to f, and we still get an SCR-diagram. □

Note. The proof shows that local confluence of α is an unnecessarily restrictive condition. In the first diagram of Fig. 8 one can replace the α-labels in the conclusion by $\mathscr{A}$ without affecting the proof.

5. Conclusion

In this paper we have presented a new confluence criterion, DCR (decreasing Church–Rosser), and we have shown that several other confluence criteria can be reduced to this one by simple transformations. There are certainly other criteria which can be reduced to local decreasingness, but the ones we have presented should give an idea of the kind of transformations involved and illustrate the power of the method.

All the confluent ARSs in this paper are in fact DCR, i.e. their confluence can be shown by choosing a suitable presentation which is locally decreasing. An interesting question is whether this holds in general, that is, whether the implication CR $\Rightarrow$ DCR holds.

A severe limitation of the Knuth–Bendix completion algorithm in the field of term rewriting systems is the fact that it is based on Newman's lemma; which requires the rewriting system to be strongly normalising. Because Newman's lemma can be viewed as a special case of our main result, which itself does not require strong normalisation of the reduction relations, it seems worthwhile to investigate whether we can use this fact to extend the completion procedure to term rewriting systems which are not strongly normalising.

Appendix A

Although finite multisets are omnipresent in computer science literature, general multisets seem to be "folklore". In this appendix we give the formal definitions of the operations on multisets and state some properties (without proof) of their interrelations (implicitly) needed in the proofs in the paper. First we give the formal definition of multisets.

Definition A.1 (*Multiset*). (1) Let $(m\in)$ $\mathbb{N}$ denote the set of natural numbers. Let $\mathbb{N}_\infty$ be $\mathbb{N}$ extended with a new element ∞. The element ∞ is the top element of the natural ordering $<$ on $\mathbb{N}$ extended to $\mathbb{N}_\infty$. The operations minimum $(\wedge)$, maximum $(\vee)$, addition $(+)$ and cutoff-subtraction $(\dot{-})$ on $\mathbb{N}_\infty$ are defined as for $\mathbb{N}$, extended as specified in Table 1.

(2) (a) A (*general*) *multiset* M over I is a map $M: I \to \mathbb{N}_\infty$.

(b) A *finite* multiset F over I is a multiset such that $\sum_{\alpha\in I} F(\alpha) < \infty$.

(c) A *set* S over I is a multiset such that $\forall \alpha \in I$, $S(\alpha) \in \{0, \infty\}$.

Multiset membership is defined by $\alpha \in M =^{\text{def}} M(\alpha) > 0$.

Multiset inclusion is defined by $M \subseteq N =^{\text{def}} \forall \alpha \in I$, $M(\alpha) \leqslant N(\alpha)$.

Definition A.2 (*Operations on multisets*).

(1) The *empty* multiset $\emptyset$ is the constant 0 function.

(2) The (finite) multiset $[\alpha]$ has value 1 at α and 0 elsewhere. The set $\{\alpha\}$ has value ∞ at α and 0 elsewhere.

(3) The binary operations *intersection* $(\cap)$, *union* $(\cup)$, *sum* $(\uplus)$ and *difference* $(-)$ are defined by: for all α in I, $(M \circledast N)(\alpha) =^{\text{def}} M(\alpha) * N(\alpha)$ via the correspondence in Table 1.

Lemma A.3 (Properties of operations). (1) *Intersection and union constitute a distributive lattice.*

Table 1
Multiset operations

$*$	$\infty * m$	$m * \infty$	$\infty * \infty$	$\circledast$
$\wedge$	m	m	∞	$\cap$
$\vee$	∞	∞	∞	$\cup$
$+$	∞	∞	∞	$\uplus$
$\dot{-}$	∞	0	0	$-$

(2) *Sum is commutative and associative. It has $\emptyset$ as neutral element.*
(3) *Sum distributes over intersection.*
(4) $S \cap (M \uplus N) = (S \cap M) \uplus (S \cap N)$.
(5) $M \cap (N - S) = (M \cap N) - (M \cap S)$.
(6) $(M \cap N) - X = (M - X) \cap (N - X)$.
(7) $(S \uplus M) - N = (S - N) \uplus (M - N)$.
(8) $(M \uplus N) - S = (M - S) \uplus (N - S)$.
(9) $(M - N) - X = M - (N \uplus X)$.
(10) $M = (M \cap N) \uplus (M - N)$.
(11) $(M - N) \cap S = (M \cap S) - N$.

Proof. The properties are easily verified by checking the corresponding properties for the extended natural numbers. □

Acknowledgment

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