Remarks on the full parallel innermost strategy

Vincent van Oostrom
Barendrecht, The Netherlands (http://www.javakade.nl)

Abstract

We make some observations on how innermost $\rightarrow_i$, parallel innermost $\mathrel{-\!\!\Vert\!\!\to}_i$ and full parallel innermost rewriting $\mathrel{-\!\!\blacksquare\!\!\to}_i$ relate for first-order term rewrite systems (TRSs).

2012 ACM Subject Classification Theory of computation $\rightarrow$ Equational logic and rewriting

Keywords and phrases TRS, innermost, full parallel innermost, confluence, termination,

Acknowledgements This note (1-6-2023) originated with questions raised by Carsten Fuhs. It is

Confluence We only employ basic concepts in abstract and first-order term rewriting [5].

▶ **Lemma 1.** *Let $\rightarrow, \hookrightarrow$ be rewrite systems on the same set of objects such that (i) $\hookrightarrow \subseteq \rightarrow^+$; and (ii) $\rightarrow \subseteq \hookrightarrow \cdot {}^{=}\!\!\leftarrow$. Then confluence of $\hookrightarrow$ entails confluence of $\rightarrow$ if (iii) $\rightarrow^= \cdot \hookrightarrow \subseteq \hookrightarrow \cdot \rightarrow^=$, and is equivalent to it if (iv) $\rightarrow$ is terminating.*

Proof. Let $\rightarrow, \hookrightarrow$ be rewrite systems on a set of objects, satisfying assumptions (i) and (ii). The assumptions allow us to speak just of normal forms as $\rightarrow$- and $\hookrightarrow$-normal forms coincide.

We first show confluence of $\hookrightarrow$ entails confluence of $\rightarrow$ assuming (iii). It suffices [5, Prop. 1.1.11] that $\lhook\joinrel\twoheadrightarrow \cdot \rightarrow^=$ has the diamond property, as the 1st inclusion in $\rightarrow \subseteq \lhook\joinrel\twoheadrightarrow \cdot \rightarrow^= \subseteq \twoheadrightarrow$ holds by reflexivity of $\lhook\joinrel\twoheadrightarrow$ and the 2nd by assumption (i). We conclude by ${}^{=}\!\!\leftarrow \cdot \twoheadleftarrow\joinrel\rhook \cdot \lhook\joinrel\twoheadrightarrow \cdot \rightarrow^= \subseteq^{(\mathrm{ii})} \rightarrow^= \cdot \twoheadleftarrow\joinrel\rhook \cdot \lhook\joinrel\twoheadrightarrow \cdot {}^{=}\!\!\leftarrow \subseteq^{\mathrm{CR}(\hookrightarrow)} \rightarrow^= \cdot \lhook\joinrel\twoheadrightarrow \cdot \twoheadleftarrow\joinrel\rhook \cdot {}^{=}\!\!\leftarrow \subseteq^{(\mathrm{iii})} \lhook\joinrel\twoheadrightarrow \cdot \rightarrow^= \cdot {}^{=}\!\!\leftarrow \cdot \twoheadleftarrow\joinrel\rhook$.

Next we show $\hookrightarrow$ is confluent iff $\rightarrow$ is, assuming (iv).

For the only–if-direction, we claim $a{\downarrow} = b{\downarrow}$ for all $a \twoheadrightarrow b$, where the normal forms $a{\downarrow}$ and $b{\downarrow}$ of a and b exist uniquely by termination (assumptions (iv) and (i)) and confluence (assumption) of $\hookrightarrow$. The claim entails confluence of $\rightarrow$ since $b \twoheadleftarrow a \twoheadrightarrow c$ gives $\hat{b} \twoheadleftarrow b \twoheadleftarrow a \twoheadrightarrow c \twoheadrightarrow \hat{c}$ for normal forms $\hat{b} = \hat{c}$ of b and c, existing by assumption (iv) and equal as $\hat{b} = \hat{b}{\downarrow} = a{\downarrow} = \hat{c}{\downarrow} = \hat{c}$ by the claim. We show the claim by well-founded induction on a w.r.t. $\leftarrow$. It being trivial for normal forms, suppose $a \rightarrow a' \twoheadrightarrow b$. Then $a \hookrightarrow b' \,{}^{=}\!\!\leftarrow a'$ for some b' by assumption (ii) and we conclude to $a{\downarrow} = b'{\downarrow} = a'{\downarrow} = b{\downarrow}$ by $a \hookrightarrow b'$ and the IH for $a' \twoheadrightarrow b'$ and $a' \twoheadrightarrow b$.

The if-direction holds since if $b \twoheadleftarrow\joinrel\rhook a \lhook\joinrel\twoheadrightarrow c$ then $\hat{b} \twoheadleftarrow\joinrel\rhook b \twoheadleftarrow\joinrel\rhook a \lhook\joinrel\twoheadrightarrow c \lhook\joinrel\twoheadrightarrow \hat{c}$ for normal forms $\hat{b} = \hat{c}$ of b and c, existing by assumptions (iv) and (i), and equal since then $\hat{b} \twoheadleftarrow a \twoheadrightarrow \hat{c}$ by assumption (i) and $\hat{b}, \hat{c}$ are normal forms, equal by the assumed confluence of $\rightarrow$. ◀

▶ **Theorem 2.** *$\rightarrow_i$ is confluent if $\mathrel{-\!\!\blacksquare\!\!\to}_i$ is, and the converse holds if $\rightarrow$ is terminating, for $\rightarrow i$ the* innermost*, cf. [1, Rem. 1] and $\mathrel{-\!\!\blacksquare\!\!\to}_i$ the* full parallel innermost *strategies of a TRS, with $\mathrel{-\!\!\blacksquare\!\!\to}_i$ defined as the* full *strategy for the (non-empty, i.e. contracting at least* 1 *redex) parallel innermost strategy $\mathrel{-\!\!\Vert\!\!\to}_i$ [5], contracting the* full *(i.e.* maximal*) set of innermost redexes.*[1]

Proof. We claim the respective assumptions of Lemma 1 hold for $\rightarrow := \mathrel{-\!\!\Vert\!\!\to}_i$ (non-empty) and $\hookrightarrow := \mathrel{-\!\!\blacksquare\!\!\to}_i$. We then conclude by the lemma since confluence of $\mathrel{-\!\!\Vert\!\!\to}_i$ and $\rightarrow_i$ coincide by $\rightarrow_i \subseteq \mathrel{-\!\!\Vert\!\!\to}_i \subseteq \twoheadrightarrow_i$. We prove the claim. (i) holds by $\mathrel{-\!\!\blacksquare\!\!\to}_i$ being a special case of $\mathrel{-\!\!\Vert\!\!\to}_i$; (ii) holds

---

[1] The notation should suggest that $\mathrel{-\!\!\blacksquare\!\!\to}$ is a *full* version of $\mathrel{-\!\!\Vert\!\!\to}$, in the same way that *full* multisteps $\mathrel{-\!\!\bullet\!\!\to}$ are a full version of multisteps $\mathrel{-\!\!\circ\!\!\to}$, contracting a maximal set of (non-overlapping) redex-patterns [2]. The analogy goes (much) further, cf. [5, Sect. 8.7]. E.g. just like $\mathrel{-\!\!\bullet\!\!\to}$ is deterministic for TRSs without *critical pairs*, $\mathrel{-\!\!\blacksquare\!\!\to}$ is deterministic for systems without *overlay* critical pairs.

since if $t \mathrel{+\!\!\!+\!\!\!\!\to}_{i,P} s$ with P its set of (pairwise parallel) positions of contracted redexes, then $s \mathrel{+\!\!\!+\!\!\!\!\to}_{i,T-P} u$ and $t \mathrel{\blacksquare\!\!\!\!\to}_i u$, obtained by contracting (in arbitrary ways) in s the innermost redexes of t at positions not in P (still innermost redex-positions in s);[2] (iii) holds since $t \mathrel{+\!\!\!+\!\!\!\!\to}_i s \mathrel{\blacksquare\!\!\!\!\to}_i u$ means $t \mathrel{+\!\!\!+\!\!\!\!\to}_{i,P} s \mathrel{+\!\!\!+\!\!\!\!\to}_{i,S} u$ for some $P \subseteq T$. If $P = T$ we conclude; otherwise the consecutive parallel steps constitute a *loath* pair [3, Sect. 4]: the innermost redexes contracted in $s \mathrel{+\!\!\!+\!\!\!\!\to}_{i,S} u$ at positions in $T - P$ can be permuted up front into (as residuals of innermost redexes in t not contracted in) $t \mathrel{+\!\!\!+\!\!\!\!\to}_{i,P} s$ giving $t \mathrel{+\!\!\!+\!\!\!\!\to}_{i,T} s' \mathrel{+\!\!\!+\!\!\!\!\to}_{i,S-(T-P)} u$; (iv) if $\to$ is terminating, then so is (non-empty) $\mathrel{+\!\!\!+\!\!\!\!\to}_i$ by $\mathrel{+\!\!\!+\!\!\!\!\to}_i \subseteq \to^+$. ◀

The theorem allows to reduce the study of confluence of full parallel innermost rewriting $\mathrel{\blacksquare\!\!\!\!\to}_i$ to that of more local, hence easier to analyse (qua properties), innermost rewriting $\to_i$; in part: without termination,[3] confluence of $\to_i$ need not entail confluence of $\mathrel{\blacksquare\!\!\!\!\to}_i$ due to the usual *out-of-sync* problem: for the trivially confluent TRS with rules $b \leftarrow a \to c$ and $b \leftrightarrow c$, the full parallel innermost steps $f(a,a) \mathrel{\blacksquare\!\!\!\!\to}_i f(b,c), f(b,b)$ are not $\mathrel{\blacksquare\!\!\!\!\to}_i$-joinable.
Termination of full parallel innermost rewriting follows from that of innermost rewriting since $\mathrel{\blacksquare\!\!\!\!\to}_i \subseteq \to_i^+$. The *quantitative* version of this, using the framework of [4], states that for every $\mathrel{\blacksquare\!\!\!\!\to}_i$-reduction of measure μ, there is a co-initial $\to_i$-reduction of measure ν such that $\mu \leq \nu$, measuring a $\mathrel{+\!\!\!+\!\!\!\!\to}_{i,P}$-step by $\#P$. It immediately follows from $\mathrel{+\!\!\!+\!\!\!\!\to}_{i,P} \subseteq \to_i^{\#P}$ and has the original *qualitative* statement as a consequence since it entails that if there were an infinite $\mathrel{\blacksquare\!\!\!\!\to}_i$-reduction, so with measure $\mu = \top$, there would be a co-initial $\to_i$-reduction with $\mu \leq \nu$, hence $\nu = \top$, so the $\to_i$-reduction would be infinite too.[4] To see also the converse quantative (and hence the (known) qualitative) statement holds, i.e. that for every $\to_i$-reduction from t of measure μ, there is a co-initial $\mathrel{\blacksquare\!\!\!\!\to}_i$-reduction of measure ν such that $\mu \leq \nu$, it suffices to instantiate (the statement in the proof of) [1, Thm. 5][5] with $\blacktriangleright := \triangleright := \to_i$, setting p to the successive p_i of[2] $T = \{p_1, \ldots, p_n\}$, yielding an $\to_i$-reduction of shape $t \to_{i,p_1} \ldots \to_{i,p_n} s \to_i \ldots$ with measure $\nu \geq \mu$, from which we conclude by iterating on s as then $t \mathrel{\blacksquare\!\!\!\!\to}_i s$.

The above gives a handle on also reducing (or simply relating) the study of quantitative termination of full parallel innermost rewriting (*macro* steps, in the terminology of [4]) to that of innermost rewriting (*micro* steps).[6]

References

1. V. van Oostrom. Random descent. In *RTA*, volume 4533 of *Lecture Notes in Computer Science*, pages 314–328. Springer, 2007. doi:10.1007/978-3-540-73449-9_24.
2. V. van Oostrom. Z; syntax-free developments. In N. Kobayashi, editor, *FSCD*, volume 195 of *LIPIcs*, pages 24:1–24:22, 2021. doi:10.4230/LIPIcs.FSCD.2021.24.
3. V. van Oostrom. On causal equivalence by tracing in string rewriting. In C. Grabmayer, editor, *TERMGRAPH*, volume 377 of *EPTCS*, pages 27–43, 2023. doi:10.4204/EPTCS.377.2.
4. V. van Oostrom and Y. Toyama. Normalisation by Random Descent. In *FSCD*, volume 52 of *LIPIcs*, pages 32:1–32:18, 2016. doi:10.4230/LIPIcs.FSCD.2016.32.
5. Terese. *Term Rewriting Systems*. Cambridge University Press, 2003.

[2] For a term t we denote its *full* set of innermost redex-positions by T, i.e. by capitalising the notation t.
[3] Without *normalisation*; the last part of the proof of Lemma 1 only uses *existence* of normal forms.
[4] Formally, in the framework of [4], infinite reductions are represented by finite *extended* reductions, that may have steps that *unfold* to infinite reductions.
[5] It should be easy to generalise [1, Thm. 5] to the setting of [4], i.e. generalising it from the length measure to an arbitrary one.
[6] To capture the exchange between the width (the amount of parallelism) and the length (the amount of causality) of the reductions; cf. Dilworth's Theorem.