On covering hexagons with diamonds, by rewriting*

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Abstract

We show each of the three rewrite techniques: random descent, proof orders for decreasing diagrams, and bricklaying, serves to solve the puzzle of covering hexagons with diamonds.

Consider an equiangular hexagon filled with diamonds; see Fig. 2 for four examples (ignoring the arrows in that figure for the moment). A *diamond* is obtained by the three ways in which an equilateral triangle whose edges are coloured widdershins red, green and blue can be glued to another such, along an edge of the *same* colour. We refer to these diamonds after their *missing* colour, that is, from left to right as displayed at the bottom of Fig. 1 as red, green and blue. We

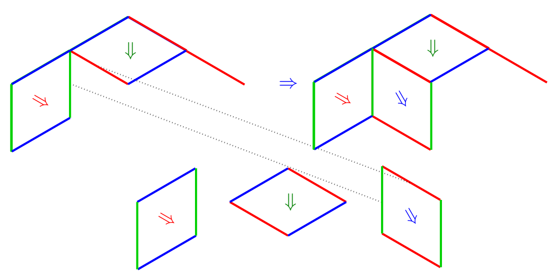

Figure 1: Filling a hexagon by repeated tiling with diamonds (⇒) of its string of -edges

show that all fillings of a given such hexagonal have the same *spectrum* [3], i.e. the same triple (r, g, b) of numbers of red r, green g, and blue b diamonds (steps), in the three ways stated in the abstract. This solves the problem of [1] for *regular* hexagons as then $r = g = b$ by symmetry.

Random descent Filling a hexagon can be seen as a transformation from its green–blue–red edges up from its bottom–leftmost node v, to its red–blue–green edges down–right from v, *sorting* edges so as to reverse the colour-order by repeated *swapping* (⇒) using the diamonds at the bottom of Fig. 1 as indicated. ‘Colouring’ [6, Ex. 7] by *measuring* [8, Def. 4] a ⇒-step as $(1, 0, 0)$, a ⇒-step as $(0, 1, 0)$, and a ⇒-step as $(0, 0, 1)$, one checks that *ordered local confluence* [6, 8] holds; the critical peak from measures $(1, 1, 1)$ either way. We conclude by [8, Lem. 24].

Proof orders for decreasing diagrams Consider the triple (r, g, b) of *area* measures [4, Ex. 3] by forgetting respectively the red, green and blue edges. For instance, for the hexagon edges in Fig. 1, we measure per [4, Fig. 5] the areas of as $(2, 2, 1)$, as $(2, 4, 2)$, and as $(2, 2, 1)$, giving the triple (of triples) $((2, 2, 1), (2, 4, 2), (2, 2, 1))$. We claim its spectrum is the triple of 2nd components $(2, 4, 2)$; cf. its fillings in Fig. 2. This holds as steps swap *adjacent* edges, so leave the areas of *those* colours unchanged but decrement that of the *other* colour. Indeed, after the two filling steps on the left in Fig. 1, we obtain with triple $((2, 1, 1)), (2, 3, 2), (2, 2, 1))$, and the further ⇒-step to the right changes only the third yielding $((2, 1, 1)), (2, 3, 2), (2, 1, 1))$. We conclude by that a string is sorted iff it has no inversions.

Remark. *The area-involutive monoid homomorphism measures the no. of inversions* per colour.

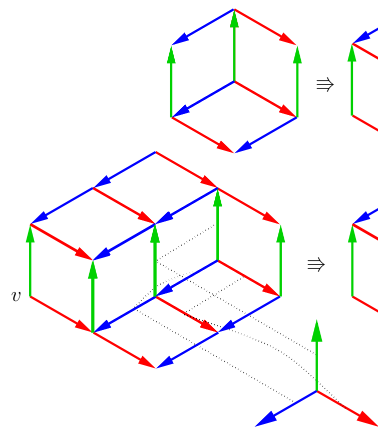

Figure 2: Proof by bricklaying

Bricklaying Consider the filling as on the left in Fig. 2, where we have directed edges to suggest an embedding in 3D. At the basis of *bricklaying* in [7], is that in any such embedding that contains a 3-*peak* (as indicated by the dotted lines), a *brick* can be laid per the ⇛-rule on top in Fig. 2; in this case not directly at the position of the 3-peak, because it has a (Blue) *in-edge*. But following that back we *must* find a 3-peak again, and now (in general: eventually) the rule *does* apply. Since ⇛-steps leave the spectrum unchanged, we conclude since from any filling the same ⇛-normal form is reached (a *brick* [7]), here after one more ⇛-step for the unique remaining 3-peak.

Acknowledgment. *Jan Willem Klop brought this problem to my attention (personal communication, 14-2-2024) explaining that he had solved it by the 'homotopic' methods of Dehornoy. After first solving it by the methods of [7], we realised it was a refinement of our 'homotopic' result [6, Ex. 2], refining counting* swaps *into counting* types of swaps *(so it generalises to arbitrarily many colours). (In hindsight, it's not surprising* random descent *[6, 8] applies since diagram filling as in [2], factors through it [9, Lem. 2].) This note illustrates the power of modern rewrite techniques.*

References

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A Remarks on equiangular hexagonals filled with diamonds

Since the diamonds (aka *rhombi* or *calissons*) used for filling (aka tiling or tessellation) are *zonogons* (centrally symmetric, convex polygons), i.e. have opposing sides of the same length, if an equiangular hexagonal can be filled by them it must be a zonogon itself, i.e. be an elongated rhombus. For instance, the equiangular hexagonal on the left in Fig. 3 is not zonogonal, hence not fillable, as is clear from colouring diamonds as in Fig. 1: the triples $(2, 2, 1)$ and $(1, 1, 2)$ of edges of each colour, going up resp. down–right from v are different, whereas such triples are preserved by filling with diamonds, by $\Rightarrow$-steps.

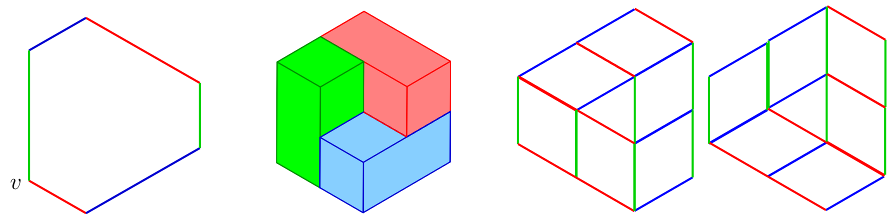

Figure 3: Non-fillable hexagon (left) and 3D (middle), non-convex non-$\Rrightarrow$-interconvertible (right)

From the perspective of sorting, filling is a *stable* sort; the (relative) order of edges of the *same* colour is preserved by filling; only edges of *distinct* colours are swapped.

Every *filled* hexagon can be realised by a series of *fillings*, i.e. filling can be sequentialised[1] into a series of $\Rightarrow$-steps rewriting the initial -segment of its boundary up from v, into the final -segment of its boundary right–down from v. This holds since the latter segment is sorted but all intermediate segments are not, so contain at least one inversion, i.e. have a location where a diamond can be filled. The same holds for filling confluence / commutation diagrams with *local* such diagrams in rewriting. We there even have [9, Rem. 2][7, Thm. 4] that a tiling of such a diagram that can be realised by tiling *vertically* by repeatedly rewriting a peak of a diagram in a family $\mathcal{D}$ of such by its valley,[2] can also be realised, see [7, Fig. 3], by tiling *horizontally* with $\mathcal{D}$, now repeatedly rewriting left legs of $\mathcal{D}$-diagrams into right legs, strengthening the known such *homotopical* results in the literature [2].[3]

The first two proofs are 2-homotopical; the third is 3-homotopical and yields the stronger result that all possible fillings are obtainable from a block by inverse $\Rrightarrow$-steps, so are all interconvertible.

In the case of hexagons, filling can even be attained by repeatedly sliding a diamond diagonally into position such that *exactly* two sides of the diamond snap into place / are glued, as suggested in Fig. 1. This holds since there always is a diamond 'on top', but may fail in general, e.g. for 2D-diagrams on cylinders (think of a cycle of toppled dominoes; no single domino can be slid upward). Similarly, in the (partial) 3D-filling in the middle in Fig. 3, none of the parts can be slid out to the *front* (or *back*; they can be slid out in each of the 4 other directions though).

In a filling as in Fig. 2 each grid-point may be classified as being an *i-peak* for i the number of *out-edges*. By the constraints, $0 \leq i \leq 3$. If all grid-points are $\leq$2-peaks, then we have a *block*-filling generalising that at the top–right of Fig. 2 [7]. E.g. at the top we *must* have a big green diamond composed of smaller such: this holds since if a -peak were filled not by a green diamond, it would be filled by a pair of red and blue diamonds, yielding another -peak. That *must* be filled itself, by it not being convex, but it couldn't be filled with a green diamond since then we would have a 3-peak. Hence, by induction this is impossible. (Convexity is necessary for this argument as witnessed by Fig. 3 right from [3], exhibiting distinct fillings in $\Rrightarrow$-normal form.)

[1] Much like how a proof net can be sequentialised into a sequent proof.

[2] Both peaks and valleys are allowed to be *degenerate* in that they might comprise one or two empty reductions.

[3] The first such homotopical result we know of goes back all the way to Sec. 6 of Newman's trailblazing paper [5], where he showed the following strengthening of the lemma nowadays simply known as *Newman's Lemma* in rewriting: a locally confluent and terminating rewrite system is confluent *and all conversions between two given objects are* deformable *into each other*, on the basis of local confluence diagrams; in particular, all conversion-cycles are contractible.