Random Descent

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Abstract. We introduce a method for establishing that a reduction strategy is normalising and minimal, or dually, that it is perpetual and maximal, in the setting of abstract rewriting. While being complete, the method allows to reduce these global properties to the verification of local diagrams. We show its usefulness both by giving uniform proofs of some known results and by establishing new ones.

1 Introduction

Consider the following two natural combinations of properties of strategies.

- normalisation (constructs a reduction to normal form from an object if it exists) and minimality (normal forms are reached in the minimal number of steps).
- perpetuality (constructs an infinite reduction from an object if it exists) and maximality (normal forms are reached in the maximal number of steps).

The former combination is of interest when implementing a rewrite system, whereas the latter is useful for its complexity analysis. Although both have received attention for various concrete strategies and rewrite systems, to our knowledge no general proof method for establishing them has been developed.

In this paper we develop methods for comparing strategies $\triangleright$ and $\blacktriangleright$ in two ways, the former being more appropriate for rewrite systems having unique normal forms, the latter for systems where normal forms need not be unique:

($\forall\forall$) For every pair of maximal $\triangleright$- and $\blacktriangleright$-reductions from the same object, the length of the first does not exceed that of the second.

($\forall\exists$) For every maximal $\triangleright$-reduction from an object, there exists a maximal $\blacktriangleright$-reduction from that object which is at least as long.

The main contributions of this paper are firstly, the reduction of both combinations of properties of the first paragraph to the ($\forall\forall$)-comparison problem, and secondly the further reduction of both comparison problems which are global (quantifying over all reductions from an object), to local properties (quanfifying only over all steps from an object).

To illustrate the power of our methods, we use examples and results from the literature. Other than that, we assume only basic rewriting knowledge.

2 Abstract rewrite systems and strategies

We recapitulate from [1, Chs. 8 and 9] the main notions from rewriting employed, *introduce* some new ones, and fix our notations. For background, motivation, and further pointers to the literature, we refer the reader to the mentioned chapters, as both our fundamental notions of ARS and strategy (going back at least to [2,3]) seem to be missing from other textbooks on rewriting.

Definition 1. *An abstract rewrite system (ARS) is a system consisting of a set of objects, a set of steps, and source and target functions mapping steps to objects [1, Def. 8.2.2].*

We employ arrow-like notations to denote ARSs, e.g. $\to$, $\triangleright$, $\blacktriangleright$, $\rightarrowtail$. That ϕ is a step of $\to$ from a to b, i.e. having a as source and b as target, is denoted by $\phi{:}a \to b$ or $a \to_\phi b$. We may omit the step, the source, or the target from this notation when irrelevant, e.g. $a \to_\phi$ indicates ϕ is a step from a, and $a \to b$ that there is a step from a to b. A normal form is an object which is not the source of a step. An object is deterministic if it is the source of at most one step. An ARS is deterministic if all objects are. We say $\blacktriangleright$ is a sub-ARS of $\to$, denoted by $\blacktriangleright \subseteq \to$, if the set of objects/steps of $\blacktriangleright$ is a subset of the set of objects/steps of $\to$, and the domain/source map of $\blacktriangleright$ is the restriction of that of $\to$.

Remark 1. We follow [2] in employing the intensional notion of abstract rewrite *system*, instead of the extensional notion of rewrite *relation*. Whereas the former allows for distinct steps between the same two objects, the latter does not. For instance, in the abstract rewrite *system* generated by the rule $I(x) \to x$ there are two distinct steps from $I(I(a))$ to $I(a)$, one corresponding to contracting the outer redex, the other to contracting the inner redex, whereas in the rewrite *relation* these are (necessarily) confounded. As a consequence, strategies such as the innermost strategy could not be expressed faithfully at the abstract level if we were to employ rewrite relations.

Using the above we present the main notion of this paper, that of a strategy.

Definition 2. *A strategy for an ARS $\to$ is a sub-ARS of $\to$ having the same sets of objects and normal forms [1, Def. 9.1.1].*

The idea is that a strategy corresponds to making a choice among the steps possible at each object. The choice may leave all possibilities open ($\to$ is a strategy for itself), but not decline all (as that would create normal forms).

Example 1. There are exactly three strategies for the ARS $a \leftrightarrows b \to c$: the ARS itself, $a \to b \to c$, and $a \leftrightarrows b \quad c$ [1, Exc. 9.1.3]. Note that e.g. the sub-ARS $a \leftarrow b \to c$ is not a strategy as it turns a into a normal form.

Remark 2. Our notion of strategy is the intensional version of the extensional notion in [3]. Like there, we do not impose additional conditions such as determinism or computability often found in the literature. Determinism would preclude expressing *the* (as opposed to *an*) innermost strategy. Computability would preclude e.g. the internal needed strategy below from being a strategy.

Henceforth $\triangleright$ and $\blacktriangleright$ are assumed to be strategies for $\to$.

As the notions of $\triangleright$-, $\to$-, and $\blacktriangleright$-normal form are all the same, we simply speak of normal forms. Since strategies are ARSs themselves, all ARS notions and the ARS constructions below apply to them. The converse of an ARS is obtained by swapping the source and target of each step, and denoted by the mirror-image of its notation, e.g. $\leftarrow$ denotes the converse of $\to$. The union of two ARSs is obtained by taking unions componentwise, and denoted by the union of the notations, e.g. $\leftrightarrow$ denotes the union of $\leftarrow$ and $\to$. A reduction is either finite or infinite. A finite reduction from a to b is inductively defined as being either the empty reduction a and then $a = b$, or a step from a to c followed by a finite reduction from c to b, for some c. An *infinite* reduction from a is coinductively defined as a step from a to b followed by an infinite reduction from b, for some b. The reduction ARS, generated by taking the finite reductions as steps, is denoted by the repetition of the notation of the original ARS, e.g. $\to$ generates $\twoheadrightarrow$. (If the repetition would lead to clutter, we affix a superscripted $*$ instead, e.g. $\leftrightarrow$ generates $\leftrightarrow^*$.) Concatenating a finite reduction $\mathcal{R}$ and a reduction $\mathcal{S}$ is defined in the usual way by induction on $\mathcal{R}$ and denoted by $\mathcal{R}\cdot\mathcal{S}$. The length of a reduction, obtained by counting the steps in it, is either finite (a natural number) or infinite (ω). A reduction is maximal if it either is a finite reduction to normal form or infinite. An object is terminating if it only allows finite reductions. An ARS is terminating, if all objects are. We will indicate (constraints on) the length of a reduction by superscripting, e.g. $\mathcal{R}{:}a \to^{\leq 5} b$ indicates that $\mathcal{R}$ is a reduction of length at most 5 from a to b, and $\mathcal{S}{:}a \to^{\omega}$ that $\mathcal{S}$ is an infinite reduction from a. A conversion is a finite $\leftrightarrow$-reduction, and we call the generated $\leftrightarrow^*$ the conversion ARS. The *distance* $d(\mathcal{R})$ of a conversion $\mathcal{R}$ is the number (an integer) of $\to$-steps minus the number of $\leftarrow$-steps in $\mathcal{R}$. An ARS is said to have unique normal forms if every object has a conversion to at most one normal form.

Remark 3. It would be interesting to extend our results below to reductions of transfinite length (applicable to concrete systems as those in [1, Ch. 12]). That would require developing an intensional version of transfinite ARSs first.

We conclude these preliminaries with formalizing the properties of strategies we would like to establish, as discussed in the first paragraph of the introduction, We already use $\triangleright$ and $\blacktriangleright$ according to the rôle they will play below.

Definition 3.
- *$\triangleright$ is normalising, if every object from which there is an $\to$-reduction to normal form, only allows finite maximal $\triangleright$-reductions.*
- *$\triangleright$ is minimal, if the length of any $\triangleright$-reduction from an object to normal form, is minimal among the $\to$-reductions from the former to the latter.*
- *$\blacktriangleright$ is perpetual, if every object from which there is an infinite $\to$-reduction, only allows infinite maximal $\blacktriangleright$-reductions.*
- *$\blacktriangleright$ is maximal, if the length of any $\blacktriangleright$-reduction from an object to normal form, is maximal among the $\to$-reductions from the former to the latter.*

3 Comparing strategies universally

We introduce a method to compare strategies $\triangleright$ and $\blacktriangleright$ for a rewrite system $\to$.

Definition 4. *$\triangleright$ is* universally better *than $\blacktriangleright$, if for every object a, and every pair of maximal $\triangleright$- and $\blacktriangleright$-reductions from a, the length of the first does not exceed that of the second.*

As we use the adverb 'universally' only to distinguish the current notion of better from the one to be introduced in the next section, we will elide it in the rest of the present section. Being better than is transitive, but not an order.

Example 2. Setting both $\triangleright$ and $\blacktriangleright$ to the ARS $a \to b \to c$ shows a strategy need not be *self*-better, i.e. better than itself. The reason for failure is that there are reductions of distinct lengths from the object a to its normal form c.

Letting $\triangleright$ and $\blacktriangleright$ be obtained from the ARS $a \to b_i \to c$ with $i \in \{1,2\}$, by omitting either of the steps $a \to b_i$, shows failure of anti-symmetry: the strategies are distinct but each is better than the other.

Removing the step from a to b in the first part of the example yields a strategy which is both better and self-better (cf. Theorem 3). These exist in general:

Proposition 1. *Each ARS has a better strategy which is self-better.*

Proof. For an ARS $\to$, let WN_i be $\{a | i = \mu n.a \to^n \cdot \not\to\}$, the set of objects whose shortest reduction to normal form has length i. The strategy $\triangleright$ is obtained from $\to$ by omitting all steps from WN_{1+i} to the complement of WN_i. As by definition each object in WN_{1+i} has *some* step to WN_i, $\triangleright$ is a strategy for $\to$. It is better than both $\to$ and itself since every maximal $\triangleright$-reduction from an object in WN_i has length i, and the other objects only allow infinite maximal reductions. □

When applied to the ARS in the second part of Example 2 the construction yields the ARS itself. More generally, it yields the largest better strategy which is self-better. Note that, dually, a self-better strategy $\blacktriangleright$ for $\to$ with $\to$ better than $\blacktriangleright$, exists, but only for $\to$ finitely branching (FB). Leaving to future research a more thorough study of the better relation, we proceed by linking it to the two combinations of properties in the first paragraph of the introduction.

Theorem 1. *If $\triangleright$ is better than $\blacktriangleright$, then:*

- *$\triangleright$ is normalising and minimal, in case $\blacktriangleright = \to$.*
- *$\blacktriangleright$ is perpetual and maximal, in case $\to = \triangleright$.*

The reverse implication holds in case $\to$ has unique normal forms.

Proof. Let $\mathcal{R}$ and $\mathcal{S}$ be maximal $\triangleright$- and $\blacktriangleright$-reductions from a.

- 'Only if': Suppose $\mathcal{S}$ ends in some normal form b. By the assumption that $\triangleright$ is better than $\blacktriangleright = \to$, the length of $\mathcal{S}$ is an upper bound on the length of any $\triangleright$-reduction from a (normalisation), in particular on that of $\mathcal{R}$ (minimality). 'If': Since otherwise there is nothing to prove, suppose $\mathcal{S}$ ends in some normal form b. By normalisation of $\triangleright$, by $\blacktriangleright = \to$, and maximality of $\mathcal{R}$, then $\mathcal{R}$ must also end in some normal form, which by uniqueness of normal forms is equal to b, from which we conclude by minimality of $\triangleright$.

– 'Only if': By the assumption that $\to = \triangleright$ is better than $\blacktriangleright$, the length of $\mathcal{R}$ is a lower bound on the length of any $\blacktriangleright$-reduction from a (perpetuality), in particular on that of $\mathcal{S}$ if $\mathcal{S}$ ends in some normal form (maximality).
'If': Since otherwise there is nothing to prove, suppose $\mathcal{S}$ ends in some normal form b. By perpetuality of $\blacktriangleright$, by $\to = \triangleright$, and maximality of $\mathcal{R}$, then $\mathcal{R}$ must also end in some normal form, which by uniqueness of normal forms is equal to b, from which we conclude by maximality of $\blacktriangleright$.

Remark 4. That the reverse implication needs uniqueness of normal forms to hold, could indicate that our notion of being better is 'too universal'; even lengths of reductions to distinct normal forms are compared. It does not seem to make much sense in general to do so (think of an ARS modelling a non-deterministic choice between otherwise unrelated computations). We leave it to future research to investigate relativising comparisons to the result (normal form) computed.

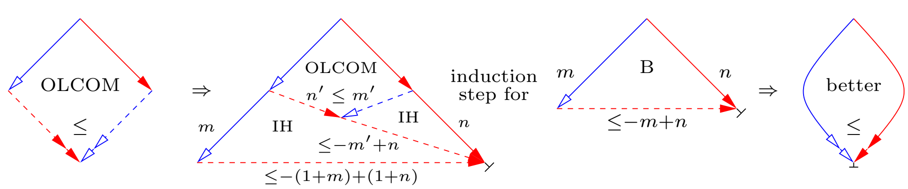
The theorem suggests that for establishing either of the combinations, normalisation and minimality or perpetuality and maximality, a single proof method for establishing that one strategy is better than another might suffice. It does.

Definition 5. *If $a \triangleleft \cdot \blacktriangleright b$ implies either $\triangleleft^\omega b$ or $a \blacktriangleright^n \cdot \triangleleft^m b$ for some $n \leq m$, then $\triangleright$* ordered locally commutes *with $\blacktriangleright$, abbreviated to OLCOM$(\triangleright, \blacktriangleright)$.*

Our reduction of the global property of 'being better' to the local property OLCOM is analogous to the way in which Newman's Lemma reduces the global property of confluence to the local property of local confluence [2] (more accurately, to the reduction of confluence to local decreasingness [4], as that method is complete). Indeed, OLCOM can be viewed as obtained from local confluence (more precisely, local commutation) by enriching the latter with an ordering constraint on the lengths of the reductions: the length of the 'left-reduction' does not exceed the length of the 'right-reduction'. Note that gluing two such diagrams together in the usual way, yields a diagram again satisfying the ordering constraint. We leave the study of these diagrams to future research. Below we will treat the left disjunct of OLCOM, i.e. the case $\triangleleft^\omega b$, as a 'degenerate' case.

Theorem 2. *OLCOM$(\triangleright, \blacktriangleright)$ only if $\triangleright$ is better than $\blacktriangleright$. The reverse holds if $\triangleright$ or $\blacktriangleright$ is equal to $\to$ and $\to$ has unique normal forms.*

Proof. 'Only if': We successively show the two implications OLCOM$(\triangleright, \blacktriangleright) \Rightarrow$ B$(\triangleright, \blacktriangleright)$, $\triangleright$ is *bounded* by $\blacktriangleright$, and B$(\triangleright, \blacktriangleright) \Rightarrow \triangleright$ is better than $\blacktriangleright$. In a diagram:



Here B$(\triangleright, \blacktriangleright)$ is defined as: for each $b \triangleleft^m a \blacktriangleright^n c$ with c in normal form, $b \blacktriangleright^{\leq -m+n} c$. We show it holds by induction on n, assuming OLCOM$(\triangleright, \blacktriangleright)$. If $m = 0$, it is trivial. Otherwise, $a \neq c$ as $\blacktriangleright$,$\triangleright$-normal forms coincide, so $b \triangleleft^m b' \triangleleft a \blacktriangleright c' \blacktriangleright^n c$. By OLCOM($\triangleright, \blacktriangleright$) either $\triangleleft^\omega c'$ or $b' \blacktriangleright^{n'} d \triangleleft^{m'} c'$ for some d and $n' \leq m'$. The former cannot hold as it would ential $d' \triangleleft^{n+1} c' \blacktriangleright^n c$ for some d', so by the induction hypothesis $d' \blacktriangleright^{-1} c$, which cannot be. In case of the latter the induction hypothesis can be applied to $d \triangleleft^{m'} c' \blacktriangleright^n c$, yielding a reduction $d \blacktriangleright^k c$ with $k \leq -m' + n$. Since $n' + k \leq n' + -m' + n \leq n$, the induction hypothesis can be applied to $b \triangleleft^m b' \blacktriangleright^{n'} d \blacktriangleright^k c$, yielding a reduction $b \blacktriangleright\!\!\blacktriangleright c$ bounded in length by $-m + n' + k \leq -m + n = -(1+m) + (1+n)$.

To show the second implication, let $\mathcal{R}$, $\mathcal{S}$ be maximal $\triangleright$-, $\blacktriangleright$-reductions from a. If $\mathcal{S}$ is infinite, there is nothing to prove. Otherwise, it is finite and ends by maximality in some normal form, say it has length n and ends in c. Then n is an upper bound on the length of $\mathcal{R}$, since the length of a mediating reduction as required by B($\triangleright, \blacktriangleright$) can only be non-negative, and by maximality $\mathcal{R}$ ends in c.

'If': Let $b \triangleleft a \blacktriangleright c$ and split cases depending on whether $\rightarrow = \triangleright$ or $\blacktriangleright = \rightarrow$.

If $\rightarrow = \triangleright$, let $\mathcal{R}$ and $\mathcal{S}$ be obtained by extending $a \triangleright b$ and $a \blacktriangleright c$ by maximal $\blacktriangleright$-reductions. If $\mathcal{S}$ is infinite, then the left disjunct of OLCOM($\triangleright, \blacktriangleright$) holds by viewing the $\blacktriangleright$-steps in the extension part of $\mathcal{S}$ as $\triangleright$-steps, using that $\blacktriangleright$ was assumed a strategy for $\rightarrow = \triangleright$. If $\mathcal{S}$ is finite, it ends by maximality in some normal form, say d. Now viewing the $\blacktriangleright$-steps in the extension part of $\mathcal{R}$ as $\triangleright$-steps, and using the assumption that $\triangleright$ is better than $\blacktriangleright$, yields that the length of $\mathcal{R}$ does not exceed that of $\mathcal{S}$. By maximality, $\mathcal{R}$ must end in a normal form, which must be equal to d by uniqueness of normal forms. Hence the right disjunct of OLCOM($\triangleright, \blacktriangleright$) holds via $\mathcal{R}$, $\mathcal{S}$.

If $\rightarrow = \blacktriangleright$, then we proceed dually by extending $a \triangleright b$ and $a \blacktriangleright c$ by maximal $\triangleright$-reductions, and subsequently viewing $\triangleright$-steps as $\blacktriangleright$-steps. $\square$

Combining Theorems 1 and 2 yields that OLCOM may be used to establish both normalisation and minimality as well as perpetuality and maximality for given strategies for an ARS $\rightarrow$. Apart from being the first method to reduce these global properties (universally quantifying over all reductions) to a local property (OLCOM universally quantifies only over pairs of steps), the method is even complete (it *is* applicable) in case $\rightarrow$ has unique normal forms.

We now illustrate the power of the method by giving uniform proofs of results for concrete rewrite systems from the literature, and by answering an open problem. The first two examples concern normalisation and minimality, the next two examples concern perpetuality and maximality, and the final one both.

Example 3. The innermost strategy for a TRS allows to contract only those redexes in a term which are innermost among all redexes. For instance, in the TRS with rules $a \to b$ and $d(x) \to g(x,x)$, the innermost strategy only allows to contract the a-redex in the term $d(a)$. Intuitively, the innermost strategy is *efficient* since it avoids duplication; the innermost reduction $d(a) \to d(b) \to g(b,b)$ is shorter than the non-innermost reduction $d(a) \to g(a,a) \to g(b,a) \to g(b,b)$, since contracting the d-redex first causes duplication of the a-redex. The catch is that the innermost strategy also avoids erasure; in the TRS with rules $a \to f(a)$ and $f(x) \to b$, an innermost reduction from $f(a)$ only contracts a-redexes and never reach the normal form b, whereas the latter could be reached *efficiently*, even in a single f-step, by erasing the a-redex. Combining these two ideas results in a strategy due to Khasidashvili, here denoted by $\triangleright$ and given by:

(internal needed strategy) contract an innermost redex among *needed* ones.

A redex is needed, in Huet and Lévy's sense, if it must be reduced in any reduction to normal form; cf. [1, Sect. 9.4.7]. Note erasable redexes are not needed.

How to show that $\triangleright$ is a normalising and minimal strategy for ordinary rewriting $\to$, in case of an orthogonal TRS? We must first show that $\triangleright$ is indeed a strategy for $\to$. This follows from the fact (due to Huet and Lévy) that in an orthogonal TRS, any term not in normal form contains a needed redex, hence also an innermost such (but since neededness is not computable, neither is $\triangleright$).

To show normalisation and minimality of $\triangleright$, it suffices by the theorem to show $\mathrm{OLCOM}(\triangleright, \to)$. This we show by a critical pair analysis, distinguishing cases on the relative positions p, q of the redexes contracted in $s \triangleleft_p t \to_q u$:

- $(=)$ Then the steps are the same by orthogonality, so $s = u$ and we conclude.
- $(\parallel)$ Then $s \to_q v \leftarrow_p u$, for some term v. Since at least one of the residuals of a needed redex is needed if it is not contracted itself, and by $p \parallel q$ the needed redex at p in t has a unique residual at p in u, that unique residual must therefore be needed. Since the latter is also innermost among the needed redexes in u (being needed or not was not changed for redexes below p), it holds $s \to_q v \triangleleft_p u$, from which we conclude.
- $(<)$ Then q is non-needed since p was assumed to be the position of a redex innermost among the needed ones. Consider a maximal $\triangleright$-reduction $\mathcal{R}$ extending $t \triangleright_p s$. Consider the projection $\mathcal{S}$ of $\mathcal{R}$ over the non-needed step $t \to_q u$. Then $\mathcal{S}$ is a $\triangleright$-reduction from u of exactly the same length as $\mathcal{R}$. This follows from the general theory of neededness, in particular from the fact that contracting a non-needed redex can neither erase, nor duplicate, nor create needed redexes. If $\mathcal{R}$ is infinite, then $\mathcal{S}$ is infinite as well so the left disjunct of $\mathrm{OLCOM}(\triangleright, \to)$ holds. Otherwise, $\mathcal{R}$ ends in a normal form by maximality, hence $\mathcal{S}$ being its projection ends in the same normal form, and the right disjunct of $\mathrm{OLCOM}(\triangleright, \to)$ holds.
- $(>)$ Then $s \to_q v \twoheadleftarrow u$ for some v, obtained by projecting the steps over one another. Per construction of the projection for orthogonal rewrite systems (essentially going back to Church and Rosser), $v \twoheadleftarrow u$ is in fact a reduction contracting the set of residuals of the redex at position p, which is a set of disjoint positions in the case of TRSs, that is $v \mathrel{\leftarrow\!\!\shortparallel\!\!-}_P u$. Partitioning P into sets of non-needed and needed residuals yields a decomposition of $v \mathrel{\leftarrow\!\!\shortparallel\!\!-}_P u$ as $v \mathrel{\leftarrow\!\!\shortparallel\!\!-} v' \mathrel{\triangleleft\!\!\triangleleft} u$, with the $\triangleright$-reduction being non-empty since p has at least one needed residual in u as it was not contracted in $t \to_q u$. Finally, consider a maximal $\triangleright$-reduction $\mathcal{R}$ from v'. If $\mathcal{R}$ is infinite, the left disjunct of $\mathrm{OLCOM}(\triangleright, \to)$ holds. Otherwise $\mathcal{R}$ ends in a normal form, and, as before, projecting $\mathcal{R}$ over $v \mathrel{\leftarrow\!\!\shortparallel\!\!-} v'$ yields a reduction from v of exactly the same length and ending in the same normal form, from which we conclude. $\square$

Since an orthogonal TRS is confluent, it has unique normal forms. Hence *that* normalisation and minimality of the internal needed strategy are reducible to

OLCOM($\triangleright, \to$) is clear by completeness of our method. The point of the examples is rather to show that a clear methodology for proving OLCOM suggests itself: the case-analysis required for OLCOM often resembles a *critical pair analysis*.

Remark 5. It would be interesting to have a critical pair *lemma* for strategies corresponding to OLCOM in the same way Huet's Critical Pair Lemma for TRSs corresponds to local confluence. That requires a suitable strategy formalism.

Example 4. In multi-step rewriting, denoted by $\multimap\!\!\to$, an arbitrary (non-zero) number of redexes in a term may be contracted at the same time. For instance, in the TRS with rules $a \to b$ and $d(x) \to g(x,x)$ as above, we have $d(d(a)) \multimap\!\!\to d(g(b,b))$ by contracting the inner d-redex and the a-redex at the same time. For the notion of 'contracted at the same time' to make sense, the redexes need to be consistent to each other. (E.g. what would it mean to contract both redexes at the same time in the term $f(g(a))$ in the TRS with rules $f(g(x)) \to b$ and $g(a) \to c$?) For orthogonal TRSs this is guaranteed giving a greedy strategy $\triangleright$:

(full-substitution strategy) contract *all* redexes in the term simultaneously.

How to show that being greedy is best, i.e. that $\triangleright$ is a normalising and minimal strategy for multi-step rewriting $\multimap\!\!\to$ in case of an orthogonal TRS? That $\triangleright$ is a strategy follows from orthogonality, since it guarantees that if *some* multi-step exists, contracting *all* redexes makes sense/is possible, cf. [1, Def. 4.9.5(v)].

To show normalisation and minimality of $\triangleright$, it suffices by the theorem to show OLCOM($\triangleright, \multimap\!\!\to$). If $s \triangleleft t \multimap\!\!\to u$, then the set of redexes contracted in the multi-step $t \multimap\!\!\to u$ is contained in the set of all redexes in t, by orthogonality. If the sets are the same, then $s = u$ and we are done. Otherwise, by standard residual theory, $s \leftarrow\!\!\multimap_P u$ where P is the set of residuals after $t \multimap\!\!\to u$ of the set of all redexes in t. If P is the set of all redexes in u, then in fact $s \triangleleft u$ and we are done again. Otherwise, we conclude from $s \multimap\!\!\to v \triangleleft u$, where $s \multimap\!\!\to v$ contracts the set of residuals after $s \leftarrow\!\!\multimap_P u$ of the set of all redexes in u. $\square$

The proof in Example 4 goes through for the λ-calculus (the full-substitution strategy is known there as Gross–Knuth reduction) and more generally to orthogonal higher-order pattern rewrite systems, as it only depends on a modicum of residual theory (e.g. [1, Thm. 11.6.29] in the case of higher-order rewriting).

Example 5. The outermost strategy for a rewrite system allows to contract only those redexes in a term which are outermost among all redexes. Intuitively, the outermost strategy is *inefficient* since it promotes duplication; the outermost-reduction $d(a) \to g(a,a) \to g(b,a) \to g(b,b)$ is longer than the non-outermost reduction $d(a) \to d(b) \to g(b,b)$ in the TRS with rules $a \to b$ and $d(x) \to g(x,x)$. One catch is that the outermost strategy also promotes erasure; in the TRS with rules $a \to f(a)$ and $f(x) \to b$, an outermost reduction from $f(a)$ immediately reaches the normal form b, whereas it would be infinitely more *inefficient* to repeat contracting a-redexes. Another catch is that outermost redexes may turn into non-outermost redexes; although the rightmost a-redex in $f(a,a)$ is outermost for the TRS with rules $a \to b$, $f(b,x) \to g(x,x)$, its contraction should be delayed if one strives for *inefficiency*, contracting instead the leftmost a-redex (turning the rightmost a-redex into an innermost one) first and next the f-redex. Combining these three ideas results in a strategy $\blacktriangleright$ due to Khasidashvili:

(limit strategy) contract an *external* redex which does not erase a reducible argument, otherwise recur on such an argument; cf. [1, Sect. 9.5.1].

Externality is Huet and Lévy's notion of a redex which remains outermost until contracted. In the term $f(a)$ above, the f-redex is external, but since contracting it would erase its, reducible, argument a, the limit strategy recurs on it.

That $\blacktriangleright$ is a perpetual and maximal strategy for $\to$ was originally proven by Khasidashvili for $\to$ being (generated by) an orthogonal Expression Reduction System. Here, we give a proof via OLCOM($\to$, $\blacktriangleright$) for the closely related formalism of orthogonal fully-extended second-order pattern rewrite systems, i.e. the restriction of Nipkow's higher-order rewrite systems to rules with free variables of second-order. That $\blacktriangleright$ is indeed a strategy for $\to$ holds since a term not in normal form contains an outermost redex, and (generalising Huet and Lévy's result for TRSs) at least one among the outermost redexes is external.

To show OLCOM($\to$, $\blacktriangleright$), we perform a critical pair analysis, distinguishing cases on the relative positions p, q of the redexes contracted in $s \leftarrow_p t \blacktriangleright_q u$.

($=$) Then $s = u$ since at most one left-hand side matches a term. We conclude.
($\parallel$) Then $s \to_q v \leftarrow_p u$, for some term v. The step $s \to_q v$ could only fail to be a $\blacktriangleright$-step, if contraction of q in t was due to a recursive call for some o above it, which is blocked in s. This cannot be, as the residual of o in s still erases the argument q is in. Hence $s \blacktriangleright_q v \leftarrow_p u$.
($<$) Then by definition of $\blacktriangleright$, p is a redex erasing q and $s \leftarrow_p u$.
($>$) Then by the diamond property for orthogonal multi-steps in PRSs [1, Thm. 11.6.29], $s \to_q v \mathrel{\leftarrow\!\!\!\!\circ\!-}_P u$ for some v, where P is the set of residuals of p after $t \blacktriangleright_q u$, which is non-empty since q is non-erasing. If $s \to_q v$ is non-erasing, then in fact $s \blacktriangleright_q v \mathrel{\leftarrow\!\!\!\!\circ\!-}_P u$ and we conclude since the non-empty multi-step $v \mathrel{\leftarrow\!\!\!\!\circ\!-}_P u$ develops into a non-empty reduction $v \twoheadleftarrow u$ by the Finite Developments Theorem [1, Thm. 11.5.11]. Otherwise,[1] consider a maximal $\blacktriangleright$-reduction $\mathcal{R}$ from s obtained by first reducing each argument erased by q in turn to its normal form. If this is finite, the redex at position q has become a $\blacktriangleright$-redex and we adjoin it to $\mathcal{R}$, and let v be the resulting term. Now consider performing for each descendant along $t \blacktriangleright_q u$ of each such an erased argument, the same steps as in $\mathcal{R}$, and do this according to the inside-order of these descendants in u. This guarantees that the resulting reduction has at least the same length as $\mathcal{R}$. (In the 3rd order case that may fail, see the following remark.) Thus if $\mathcal{R}$ is infinite this yields an infinite reduction from u as well, and we are done. Otherwise, this yields a reduction ending in v by [1, Thm. 11.6.29] and we conclude again. □

The limit strategy need not be maximal nor perpetual for third-order systems:

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[1] An example of this in λ-calculus is $(\lambda x.y)N \leftarrow (\lambda x.(\lambda z.y)x)N \blacktriangleright (\lambda z.y)N$; we first *should* $\blacktriangleright$-reduce N to normal form, say N', before to proceed with $(\lambda x,y)N' \blacktriangleright y$.

Remark 6. Consider $t = f(G.g(x.G(x)))$ in the orthogonal fully-extended third-order rewrite system:

$$a \to b \qquad g(x.G(x)) \to G(a) \qquad f(G.H(G)) \to H(y.c)$$

Starting with contracting the head-redex-pattern yields $t \blacktriangleright g(x.c) \blacktriangleright c$. Observe how contracting the head-redex makes the inside g-redex erasing (something impossible in second-order systems) and that contracting the latter first yields a longer reduction: $t \to f(G.G(a)) \to f(G.G(b)) \to c$.

Example 6. Leftmost-outermost redexes are external in $\lambda\beta$-calculus. Therefore

(F_∞) contract a leftmost-outermost redex which does not erase a reducible argument, otherwise recur on such an argument.

is a limit strategy, hence maximal and perpetual by the above; cf. [5, Thm. 11]. As on λI-terms the leftmost-outermost strategy *is* F_∞, it is maximal and perpetual; cf. [6, Proposition 3.17]. What about F_∞ for λ-calculi with explicit substitutions?

The $\lambda\mathtt{x}^-$-calculus of [7] is a $\lambda\beta$-calculus with explicit substitution operator $\langle := \rangle$, and rules (of which only the third, not the first, is erasing (N)!):

$$\begin{array}{ll} (\lambda x.M)N \to M\langle x{:=}N\rangle & \\ x\langle x{:=}N\rangle \to N & (\lambda y.M)\langle x{:=}N\rangle \to \lambda y.M\langle x{:=}N\rangle \\ y\langle x{:=}N\rangle \to y & (M_1M_2)\langle x{:=}N\rangle \to M_1\langle x{:=}N\rangle M_2\langle x{:=}N\rangle \end{array}$$

As any reducible term has a leftmost-outermost redex, the ARS $\blacktriangleright$ induced by F_∞ is a strategy for the ARS $\to$ induced by $\lambda\mathtt{x}^-$. Since $\lambda\mathtt{x}^-$ is a second-order rewriting system [8, Def. 13], to show OLCOM($\to, \blacktriangleright$) it suffices to adapt the analysis of Example 5. The only property of externality we employed there was that it is preserved for residuals (if any). As that also holds for leftmost-outermostness in $\lambda\mathtt{x}^-$, it suffices to supplement the case-analysis with the (unique) critical pair:

$$C[M\langle x{:=}N\rangle\langle y{:=}P\rangle] \leftarrow_p C[((\lambda x.M)N)\langle y{:=}P\rangle] \blacktriangleright_q C[(\lambda x.M)\langle y{:=}P\rangle N\langle y{:=}P\rangle]$$

We reason as for $(>)$ in Example 5. In particular, we simulate any $\blacktriangleright$-reduction $\mathcal{R}$ from the term s on the left by a reduction $\mathcal{S}$ from the term u on the right, which is at least as long. To that end, we first reduce u one step further to $u' = C[M\langle y{:=}P\rangle\langle x{:=}N\langle y{:=}P\rangle\rangle]$. After that, simulation of $\mathcal{R}$ by $\mathcal{S}$ is redex-wise: By definition of F_∞, a $\langle x{:=}N\rangle\langle y{:=}P\rangle$-closure is only ever distributed (over λ or @) in its entirety in $\mathcal{R}$, which can be simulated in $\mathcal{S}$ by distributing the $\langle y{:=}P\rangle\langle x{:=}N\langle y{:=}P\rangle\rangle$-closure. In case a $\langle x{:=}N\rangle\langle y{:=}P\rangle$-closure is being applied to a variable in $\mathcal{R}$, its simulation in $\mathcal{S}$ is defined by cases on the variable:

(x) Then $x\langle x{:=}N\rangle\langle y{:=}P\rangle \blacktriangleright N\langle y{:=}P\rangle \leftarrow^2 x\langle y{:=}P\rangle\langle x{:=}N\langle y{:=}P\rangle\rangle$.

(y) Then $y\langle x{:=}N\rangle\langle y{:=}P\rangle \blacktriangleright^i y\langle x{:=}N'\rangle\langle y{:=}P\rangle \blacktriangleright y\langle y{:=}P\rangle \blacktriangleright P \blacktriangleright^j P'$ where N',P' are the normal forms of N,P (if any). This can be simulated by $y\langle y{:=}P\rangle\langle x{:=}N\langle y{:=}P\rangle\rangle \to^i y\langle y{:=}P\rangle\langle x{:=}N'\langle y{:=}P\rangle\rangle \to P\langle x{:=}N'\langle y{:=}P\rangle\rangle \to^j P'\langle x{:=}N'\langle y{:=}P\rangle\rangle \to^+ P'$, using for the final reduction that neither the variable x nor any closures occur in the normal form P'.

(z) Analogous to the previous case ending in z (if at all) instead of P'. $\square$

This solves the open problem [9, Rem. 3.18]. The method can easily be adapted to show maximality and perpetuality of F_∞ for the $\lambda\beta\eta$-calculus [5, Thm. 19].[2]

Remark 7. Termination proofs for typed λ-calculi (with explicit substitutions) usually involve a 'syntactic' commutation property. The strategy F_∞ allows to factor this property out, giving rise to fully 'semantic' termination proofs.

We say $\to$ is *ordered weak Church–Rosser* (OWCR) if OLCOM($\to,\to$). OWCR entails $\to$ is self-better (Theorem 2) and if an object can be reduced to normal form in n steps, then *any* strategy will do so (Theorem 1); in Newman's terminology [2, p. 226]: the end-form is reached by random descent. Inspired by this, we say $\to$ has *random descent* (RD), if for each $\mathcal{R}{:}a \leftrightarrow^* b$ with b in normal form, all maximal reductions from a have length $d(\mathcal{R})$ and end in b. Note that for an ARS satisfying OWCR it suffices to prove normalisation to establish both termination and (random descent) confluence; see [10] for an example of this.

Theorem 3 (Random Descent). *OWCR $\Leftrightarrow$ RD.*

Proof. 'Only if': Setting $\triangleright = \to = \blacktriangleright$, we obtain OWCR $\Rightarrow$ B($\to,\to$) by using the implication OLCOM($\triangleright,\blacktriangleright$) $\Rightarrow$ B($\triangleright,\blacktriangleright$) in the proof of Theorem 2. Next, we claim B($\to,\to$) implies *self-boundedness* (SB), where SB states that for any conversion $\mathcal{R}{:}a \leftrightarrow^* b$ with b in normal form, there exists $a \to^{\leq d(\mathcal{R})} b$. The claim follows by an easy induction on the number of peaks in the conversion (analogous to the way the Church–Rosser property is proven from confluence).

Having established SB, we prove RD. Let $\mathcal{R}{:}a \leftrightarrow^* b$ with b in normal form, $n = d(\mathcal{R})$ and consider a maximal reduction $\mathcal{S}$ from a. On the one hand, n is an *upper* bound on the length of $\mathcal{S}$, since otherwise there would be a reduction $\mathcal{S}'{:}a \to^{n+1} a'$ hence a conversion to normal form $\mathcal{S}'^{-1} \cdot \mathcal{R}{:}a' \leftrightarrow^* b$ with negative distance -1, contradicting SB. Thus by maximality $\mathcal{S}$ ends in a normal form b' which by SB for $\mathcal{S}^{-1} \cdot \mathcal{R}{:}b' \leftrightarrow^* b$ reduces to b, so $b' = b$. On the other hand, n is a *lower* bound on the length of $\mathcal{S}$, as else $\mathcal{R}^{-1} \cdot \mathcal{S}{:}b \leftrightarrow^* b$ would be a conversion to normal form having negative distance, contradicting SB.

'If': Let $\mathcal{R}{:}b \leftarrow a \to c$, and let $\mathcal{S}$ be a maximal reduction from c. If $\mathcal{S}$ is infinite, we are done. Otherwise, it is finite and ends in a normal form, and we conclude from RD using $d(\mathcal{S}) = d(\mathcal{R} \cdot \mathcal{S})$. ◻

Instances of calculi for which random descent has been established and its consequences used, abound in the literature, almost invariably proven in ad hoc fashion. Some examples are linear λ-calculi [11, Cor. 3.4], [5, Prop. 33], spine strategies for λ-calculus [6, Prop. 4.21], in/external strategies for orthogonal TRSs [1], orthogonal string rewrite systems [12], orthogonal graph rewrite systems in particular Lafont's interaction nets [13], and orthogonal process calculi in particular those modelled by Stark's concurrent transition systems [14].

Several local conditions sufficient for RD, generalizing [2, Thm. 2], have been proposed in the literature, e.g. *balanced weak Church–Rosser* (BWCR [3,15]) *balanced SCR* [16], *linear biclosed* [8], and *SCR*$^{\geq 1}$ [16]. However, none is complete

[2] Only failure of preservation of leftmost-outermostness $(\lambda x.(\lambda y.Kyx)x)$ requires care.

as they all fail to cover the ARS $a \to b \to c \to d \to c \leftarrow a$, which *does* satisfy OWCR. In fact, no global property had been explicitly identified before, but it is the global property upon which all applications are seen to rely. For instance, all results in [3,15] trivially generalise by replacing everywhere BWCR by RD.

Example 7. Let the ARS $\to$ have finite lists as objects and swapping of adjacent elements in lists-which-are-not-sorted as steps. The strategy $\triangleright$, which only swaps elements which are out-of-order, so-called *inversions*, has random descent as follows from OWCR($\triangleright$), the only interesting case being the critical pair schematically given by $bca \triangleleft cba \triangleright cab$, which is completed as $bca \triangleright bac \triangleright abc \triangleleft acb \triangleleft cab$. Since insertion sort is an instance of $\triangleright$ which is $\Theta(n^2)$, and OLCOM($\triangleright, \to$) follows by an easy critical pair analysis, we conclude sorting-by-swapping is $\Omega(n^2)$.

4 Comparing strategies existentially

We introduce our second (and third) way to compare strategies.

Henceforth it is assumed that $\to \,=\, \blacktriangleright \cup \triangleright$.

Definition 6. *An* ordered pair *is a pair of maximal reductions from an object, such that if the second ends, the first ends in the same object and is not longer. $\mathcal{R}$ can be* completed *on the left (right) by $\mathcal{S}$ if $\mathcal{S}$,$\mathcal{R}$ ($\mathcal{R}$,$\mathcal{S}$) is an ordered pair.*

- $\triangleright$ *is* existentially better *than* $\blacktriangleright$*, if every maximal* $\blacktriangleright$*-reduction can be completed on the left by a* $\triangleright$*-reduction.*
- $\blacktriangleright$ *is* existentially worse *than* $\triangleright$*, if every maximal* $\triangleright$*-reduction can be completed on the right by a* $\blacktriangleright$*-reduction.*

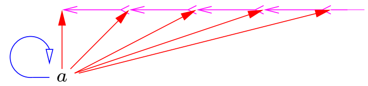
We drop de adverb ‘existentially’ if no confusion with the notion of better from the previous section can arise. Clearly, both better and worse are quasi-orders, but neither needs to be anti-symmetric (for the same reason as before).

Remark 8. For ARSs having unique normal forms, $\triangleright$ being universally better than $\blacktriangleright$ implies both $\triangleright$ being existentially better than $\blacktriangleright$ and $\blacktriangleright$ being existentially worse than $\triangleright$ but not vice versa. Since our existential ways to compare strategies are relativised with respect to the normal forms, whereas the universal way was not, they are in general incomparable for ARSs not having unique normal forms.

The next goal is to reduce the global properties better and worse to local ones.

Definition 7.
- $\to$ *has* left extraction *(LE) if every finite maximal reduction of shape $\blacktriangleright \cdot \triangleright\!\!\triangleright$ can be completed on the left by a reduction of shape $\triangleright \cdot \twoheadrightarrow$.*
- $\to$ *has* right extraction *(RE) if every finite maximal reduction of shape $\triangleright \cdot \blacktriangleright\!\!\blacktriangleright$ can be completed on the right by a reduction of shape $\blacktriangleright \cdot \twoheadrightarrow$.*

The idea for LE is that to show $\triangleright$ is better than $\to$, it suffices that any *initial* $\blacktriangleright$-step can be ‘improved’ into a $\triangleright$-step. The dual idea for RE will need extra assumptions to work, i.e. to imply that $\blacktriangleright$ is worse than $\to$, as shown by:



RE holds but $a \triangleright^{\omega}$ cannot be completed on the right by $\blacktriangleright$ ($\rightarrowtail\; = \triangleright \cap \blacktriangleright$)

Remark 9. LE, OLCOM, RE are essentially the same diagram, the main difference residing in which sides are existentially quantified (left, bottom, right).

Lemma 1. *$\rightarrow$-non-termination implies $\blacktriangleright$-non-termination, under the assumptions that $a \triangleright \cdot \blacktriangleright^n$ implies $a \blacktriangleright \cdot \rightarrow^n$ and that $\blacktriangleright$ is finitely branching (FB)*

Proof. By FB and König's Lemma, it suffices to show that for any $\rightarrow$-reduction there exists a $\blacktriangleright$-reduction from the same object and of the same length. The proof is by induction on the length of the $\rightarrow$-reduction. The base case being trivial, suppose it is of shape $a \rightarrow a' \rightarrow^n$. By the induction hypothesis for $a' \rightarrow^n$, there exists a reduction $a' \blacktriangleright^n$. If in fact $a \blacktriangleright a'$, we are done. Otherwise, $a \triangleright a'$ and the assumption for $a \triangleright a' \blacktriangleright^n$ yields $a \blacktriangleright a'' \rightarrow^n$ from which we conclude by the induction hypothesis applied to $a'' \rightarrow^n$. $\square$

Two sufficient conditions for the lemma are obtained by requiring on top of FB, either RE with 'finite' removed from its definition, or [17, Lemma 7].

Theorem 4. *If LE, then $\triangleright$ is better than $\rightarrow$. If RE, then $\blacktriangleright$ is worse than $\rightarrow$, under the assumptions of Lemma 1.*

Proof. To prove the first item, let $\mathcal{R}$ be a maximal reduction from a. If $\mathcal{R}$ is infinite, then we are done. Otherwise $\mathcal{R}$ is finite and we proceed by induction on its length. The base case being trivial, suppose $\mathcal{R}$ is of shape $a \rightarrow a' \twoheadrightarrow b$. By the induction hypothesis for $a' \twoheadrightarrow b$, we get $a' \triangleright\!\triangleright b$ which is no longer, so we are done if in fact $a \triangleright a'$. Otherwise LE for $a \blacktriangleright a' \triangleright\!\triangleright b$ yields $a \triangleright a'' \twoheadrightarrow b$ which is no longer, so by the induction hypothesis for $a'' \twoheadrightarrow b$, $a'' \triangleright\!\triangleright b$ which is no longer.

To prove the second item, note that if $\rightarrow$ is non-terminating for a, then Lemma 1 yields an infinite $\blacktriangleright$-reduction from a and we are done. Otherwise, we proceed as in the first item, but by well-founded induction ordered by $\rightarrow$ on the source a of $\mathcal{R}$, exchanging longer with shorter, LE with RE, and $\triangleright$ with $\blacktriangleright$. $\square$

As a typical application of the above, we present a λ-calculus λ^+ with *non-deterministic choice* embodied by the rule $M_1 + M_2 \rightarrow M_i$ (cf. [18]).

Lemma 2. *RE without 'finite' in its condition hold for $\blacktriangleright$ the F_∞-strategy and $\triangleright$ the reduction relation of λ^+, with F_∞ as for λ-calculus (Example 6) additionally choosing for $M_1 + M_2$ either to recur on M_i if it is reducible, or to select M_i in case the other argument is in normal form.*

Proof. Let $t \triangleright_p s \blacktriangleright^{\alpha}$ be a maximal reduction and distinguish cases on whether s is a normal form or not. If s is a normal form, we show $t \blacktriangleright_q u \twoheadrightarrow s$, for some u. If in fact $t \blacktriangleright_p s$ this is trivial. Otherwise, the step $t \triangleright_p s$ must be outermost and all other redex-patterns in t must be below p and erased by the step. Then for any $t \blacktriangleright t'$ it holds $t' \triangleright_p s$. If s is not a normal form, the reduction is of shape $t \triangleright_p s \blacktriangleright_q u \blacktriangleright^{\alpha \dot{-} 1}$ for some u. It suffices to show that this entails $t \blacktriangleright \cdot \rightarrow^+ u$. Let $t = C[P]_p$ and distinguish cases on the relative positions of p,q.

($\leq$) If $P = (\lambda x.M)N$ then either P is in fact a $\blacktriangleright$-redex, or x does not occur in M and N is not normal, thus N is erased and we proceed as above. If $P = M_1 + M_2$ and, say, M_1 is selected and then $\blacktriangleright$-rewritten to M_1', then $t \blacktriangleright C[M_1' + M_2]_p \to u$.

($\not\leq$) If $t \blacktriangleright_q$ then $t \blacktriangleright_q \cdot \twoheadrightarrow u$ by [1, Thm. 11.6.22], where in fact the $\to$-reduction cannot be empty since $\blacktriangleright$-steps cannot erase $\triangleright$-steps. Otherwise, contracting P turns the 'q-branch' of the greatest common predecessor o of p, q in the term-tree, into an 'F_∞-branch'. This can only happen if either the step was in fact a step to normal form of the 'p-branch', or if it created the body of a β-redex at position o. In either case it is a $\blacktriangleright$-step unless it erases a non-normal argument N, but then we may proceed as before. □

As any term contains only finitely many occurrences of $+$, $\triangleright$ is finitely branching, and by $\blacktriangleright \subseteq \triangleright$, $\blacktriangleright$ is so too. Thus by Theorem 4 $\blacktriangleright$ is worse than $\triangleright$. This can be useful to prove termination of typed subcalculi of λ^+ via termination of F_∞; cf. Remark 7. We conclude with a case-study of abstract copying.

Definition 8. *If $\blacktriangleright = \bigcup_p \blacktriangleright_p$, then $\blacktriangleright$* copies *$\triangleright$ if for $b \triangleleft a \blacktriangleright_p$, either $b \triangleleft\!\triangleleft \cdot \blacktriangleleft_p a$ or q exists with ($a \blacktriangleright_p a'$ implies $b \blacktriangleright_q \cdot \triangleleft^+ a'$, and $b \blacktriangleright_q b'$ implies $a \blacktriangleright_p \cdot \triangleright^+ b'$).*

E.g. the outermost strategy copies $\to_{\mathcal{T}}$ for $\mathcal{T} = \{f(x) \to g(x), f(x) \to h(x,x)\}$.

Theorem 5. *If $\blacktriangleright$ copies $\triangleright$, then $\blacktriangleright$ is worse than $\triangleright$.*

Proof. Let $a_0 \blacktriangleright_p$ and let $a_0 \triangleright a_1 \triangleright^\alpha$ be maximal. It suffices to find $a_0 \blacktriangleright b \triangleright^\beta$ which is no shorter and ends in the same normal form if at all, as repeating this on b leads to an ever growing $\blacktriangleright$-reduction from a_0, while preserving the property.

Setting $p_0 = p$, construct a maximal sequence of indices p_i such that ($a_i \blacktriangleright_{p_i} a'$ implies $a_{i+1} \blacktriangleright_{p_{i+1}} \cdot \triangleleft^+ a'$, and $a_{i+1} \blacktriangleright_{p_{i+1}} b'$ implies $a_i \blacktriangleright_{p_i} \cdot \triangleright^+ b'$). Look for the first i for which p_{i+1} is not defined.

If it exists, then a_i cannot be in normal form since $a_i \blacktriangleright_{p_i}$ holds by induction on i with base case $a_0 \blacktriangleright_{p_0}$. Hence by maximality $a_i \triangleright a_{i+1}$ implying $a_i \blacktriangleright_{p_i} b_i \triangleright\!\triangleright a_{i+1}$ for some b_i. Per construction, there exists $b_{i \dot- 1}$ such that $a_{i \dot- 1} \blacktriangleright_{p_{i \dot- 1}} b_{i \dot- 1} \triangleright^+ b_i$. Continuing in this fashion by induction on i, yields $a_0 \blacktriangleright_{p_0} b_0 (\triangleright^+)^i b_i \triangleright\!\triangleright a_{i+1} \triangleright^{\alpha \dot- i}$ which is as desired.

If it doesn't exist, then select an arbitrary step $a_0 \blacktriangleright_{p_0} b_0$. Per construction, there exists b_1 such that $a_1 \blacktriangleright_{p_1} b_1 \triangleleft^+ b_0$. Continuing in this fashion by induction, yields $a_0 \blacktriangleright_{p_0} b_0 \triangleright^+ b_1 \triangleright^+ b_2 \triangleright^\omega$ which has the desired property. □

It is easy to see that for TRSs the innermost strategy not only copies itself, but also copies, hence by Theorem 5 is worse than, non-dup-generalized innermost rewriting$\to_{ndg}$ [17], generalizing all results on $\to_{ndg}$ in [17] to the non-finitely branching case. For another application of Theorem 5: any positional [1, p. 512] innermost strategy copies the innermost strategy, hence termination of the former implies termination of the latter, simplifying [19, Thm. 6] and [20, Thm. 2].

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