Sub-Birkhoff

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Abstract. For equational specifications validity coincides with derivability in equational logic, which in turn coincides with convertibility generated by the rewrite relation. It is shown that this correspondence, essentially due to Birkhoff, can be generalised in a uniform way to sub-equational logics such as Meseguer's rewriting logic.

1 Introduction

In order to motivate and state our generalisation, we illustrate the essential ingredients of the usual correspondence (see, e.g. Chapter 7 of [1] or Chapter 3 of [2]) between validity, derivability and convertibility by means of the following equational specification $\mathcal{EM}ul$ of addition and multiplication:

$$\mathtt{A}(x,\mathtt{0}) \approx x \tag{1}$$

$$\mathtt{A}(x,\mathtt{S}(y)) \approx \mathtt{S}(\mathtt{A}(x,y)) \tag{2}$$

$$\mathtt{M}(x,\mathtt{0}) \approx \mathtt{0} \tag{3}$$

$$\mathtt{M}(x,\mathtt{S}(y)) \approx \mathtt{A}(x,\mathtt{M}(x,y)) \tag{4}$$

and the equation:

$$\mathtt{M}(\mathtt{S}(x),\mathtt{S}(\mathtt{0})) \approx \mathtt{S}(x) \tag{5}$$

On the one hand, (5) is *valid* for the specification $\mathcal{EM}ul$ in the sense that it holds in any model. In algebraic semantics, terms are giving meaning by means of an algebra. The algebra is then called a model of the specification if each equation in the latter holds in the former. That is, the meanings of the left- and right-hand side of the equation are *identical*, for any assignment to the variables. For instance, the algebra $\mathcal{N}at$ having the set of natural numbers as carrier, and interpreting $\mathtt{0}$, $\mathtt{S}$, $\mathtt{A}$ and $\mathtt{M}$ as zero, successor, addition and multiplication, respectively, is a model of $\mathcal{EM}ul$ and one easily verifies that (5) holds in it. For instance, for the assignment α mapping every variable to the natural number 2, its left-hand side $\mathtt{M}(\mathtt{S}(x),\mathtt{S}(\mathtt{0}))$ is mapped to $(2+1)\times(0+1)$, and its right-hand side $\mathtt{S}(x)$ to $2+1$, i.e. both sides are mapped to 3.

On the other hand, (5) being the conclusion of the proof tree:

$$\dfrac{\dfrac{\dfrac{}{\mathtt{M}(x,\mathtt{S}(y)) \approx \mathtt{A}(x,\mathtt{M}(x,y))}\,(4)}{\mathtt{M}(\mathtt{S}(x),\mathtt{S}(\mathtt{0})) \approx \mathtt{A}(\mathtt{S}(x),\mathtt{M}(\mathtt{S}(x),\mathtt{0}))}\,(\sigma) \qquad \dfrac{\dfrac{\dfrac{}{\mathtt{S}(x) \approx \mathtt{S}(x)}\,(\text{ref}) \quad \dfrac{\dfrac{}{\mathtt{M}(x,\mathtt{0}) \approx \mathtt{0}}\,(3)}{\mathtt{M}(\mathtt{S}(x),\mathtt{0}) \approx \mathtt{0}}\,(\sigma)}{\mathtt{A}(\mathtt{S}(x),\mathtt{M}(\mathtt{S}(x),\mathtt{0})) \approx \mathtt{A}(\mathtt{S}(x),\mathtt{0})}\,(\mathtt{A}) \quad \dfrac{\dfrac{}{\mathtt{A}(x,\mathtt{0}) \approx x}\,(1)}{\mathtt{A}(\mathtt{S}(x),\mathtt{0}) \approx \mathtt{S}(x)}\,(\sigma)}{\mathtt{A}(\mathtt{S}(x),\mathtt{M}(\mathtt{S}(x),\mathtt{0})) \approx \mathtt{S}(x)}\,(\text{trans})}{\mathtt{M}(\mathtt{S}(x),\mathtt{S}(\mathtt{0})) \approx \mathtt{S}(x)}\,(\text{trans})$$

with substitution σ such that $x \mapsto \mathtt{S}(x)$ and $y \mapsto \mathtt{0}$, shows that it is *derivable* in equational logic (see Table 1).

On the gripping hand, *convertibility* of the sides of (5) is witnessed by:

$$\underline{\mathtt{M}(\mathtt{S}(x),\mathtt{S}(\mathtt{0}))} \to \mathtt{A}(\mathtt{S}(x),\underline{\mathtt{M}(\mathtt{S}(x),\mathtt{0})}) \to \underline{\mathtt{A}(\mathtt{S}(x),\mathtt{0})} \to \mathtt{S}(x)$$

a sequence of forwards (and possibly backwards) rewrite steps.

We will refer to the correspondence between validity and derivability as *Birkhoff's theorem* since it is due to [3], and to the correspondence between derivability and convertibility as *logicality* (cf. [4]). Both correspondences are of fundamental importance in the study of programming language foundations, see [5], and can be seen as a justification of term rewriting itself. For instance, they allow for solving uniform word problems by means of complete term rewriting systems.

As argued by Meseguer, e.g. in [6], some specifications should not be considered to be equational. For instance an *equational* specification of a binary choice function ? (selecting either of its arguments) does not make sense, and would result in all terms being identified to one another. Instead an *ordering* specification is appropriate here:

$$\begin{aligned} ?(x,y) &\gtrapprox x \\ ?(x,y) &\gtrapprox y \end{aligned}$$

As suggested by the notation, in a model of such an ordering specification each left-hand side should be greater than or equal to the corresponding right-hand side. Then, to salvage the correspondence between validity and derivability, the symmetry rule (sym) should dropped from the proof system of equational logic in Table 1, resulting in *ordering*[1] logic. In order to regain the correspondence between derivability and convertibility, backwards steps should be dropped from the latter. After this is done, both correspondences hold again as shown in [6].

Here we propose to generalise the correspondence as presented above for equational and ordering logic, to so-called *sub-equational* logics obtained by dropping a subset of the inference rules of equational logic. In particular, equational and ordering logic are obtained by dropping nothing (the empty set) and the

[1] Beware: in this paper we will use a systematic naming scheme. For instance, our ordering logic is known in the literature as *rewriting* logic.

$$\frac{}{s \approx t}\,(s \approx t \in E) \quad \frac{s \approx t}{\sigma(s) \approx \sigma(t)}\,(\sigma{:}X \to T(\Sigma, X)) \quad \frac{s_1 \approx t_1 \quad \dots \quad s_n \approx t_n}{f(s_1, \dots, s_n) \approx f(t_1, \dots, t_n)}\,(f \in \Sigma)$$

$$\frac{}{s \approx s}\,(\text{ref}) \quad \frac{s \approx t}{t \approx s}\,(\text{sym}) \quad \frac{s \approx t \quad t \approx u}{s \approx u}\,(\text{trans})$$

Table 1. Equational logic

singleton $\{(\text{sym})\}$, respectively. We argue that sub-equational logics are interesting for the same reason that ordering logic is interesting [6]: enforcing too many inference rules would conflate notions which one would like to keep distinct. Just as confusing the forwards and backwards directions (as enforced by symmetry) would be a brutal [6] act in case of the (non-confluent) ordering specification for choice above, to confuse 'not being able to do anything' with 'being able to do a trivial step' (as enforced by reflexivity) is a brutal act in case of a (terminating) specification such as

$$a_{n+1} > a_n$$

Similarly, single-steps should, *a priori*, not be confused with many-steps (as would be enforced by transitivity) in case of a *step* specification (a.k.a. term rewriting system), since by the choice for that form of specification one implicitly specifies that one is interested in individual steps (think e.g. of complexity).

Based on the above ideas, we give a parametrised account of both Birkhoff's theorem (Section 3) and logicality (Section 4) for sub-equational specifications (Section 2). The proofs of our results are simple, as they are just variations on the existing simple proofs for equational logic. The main effort will be in formalising both the results and their proofs in a way which allows for their parametrisation. As a side-effect of this parametrisation the proof structure becomes clearer, which may be of some didactic value. Because of it, we have made an effort to make the paper self-contained.

2 Sub-equational specifications

A sub-equational specification can be thought of as an equational specification together with a set of inference modes specifying how its equations are to be interpreted (e.g. indeed as equations, or alternatively as rewrite rules, or ...).

Definition 1. *A* signature $(f, g, h \in)\Sigma$ *is a set of* symbols*, each of which comes equipped with a natural number* arity*.*

The subset of Σ consisting of all symbols of arity n, is denoted by $\Sigma^{(n)}$. Elements of $\Sigma^{(0)}$ are called *constants*. Throughout, we assume $(x, y, z \in)X$ to be a signature disjoint from Σ, consisting of an infinite number of constants called *variables*.

Definition 2. *The set* $(s, t, u \in)T(\Sigma)$ *of* Σ-terms *is inductively defined by:*

– $fs_1 \dots s_n$ *is a term, if* f *is an* n*-ary symbol and* $s_1, \dots, s_n$ *are terms.*

The set $T(\Sigma, X)$ *of* Σ-terms over X *is defined as* $T(\Sigma \cup X)$.

As is customary, we may write $f(s_1, \dots, s_n)$ to denote $fs_1 \dots s_n$.

Example 1. Consider the signature Σ consisting of the nullary symbol $\mathtt{0}$, the unary symbol $\mathtt{S}$ and the binary symbols $\mathtt{A}$ and $\mathtt{M}$. Some Σ-terms are $\mathtt{0}$, $\mathtt{S0}$, $\mathtt{SS0}$, $\mathtt{A00}$ and $\mathtt{MA0S00}$. E.g. the last term is also denoted by $\mathtt{M}(\mathtt{A}(\mathtt{0}, \mathtt{S}(\mathtt{0})), \mathtt{0})$. An example of a Σ-term over X is $\mathtt{A}(x, \mathtt{S}(y))$.

Definition 3. *A* Σ-statement *is a pair of* Σ*-terms.*

Definition 4. *A* (sub-equational) specification *is a quadruple* $\mathcal{S} := \langle \Sigma, X, S, L \rangle$ *with* S *a set of* $\Sigma \cup X$*-statements and* L *a set of* inference modes *which is a subset of* {(embedding), (compatibility), (reflexivity), (symmetry), (transitivity)}.

We will abbreviate the respective inference modes to (emb), (comp), (ref), (sym), and (trans). The idea is that for a sub-equational specification $\mathcal{S} := \langle \Sigma, X, S, L \rangle$, the modes of inference will specify how the pairs in S are to be dealt with, both at the semantical and the syntactical level (both to be presented below).

Example 2. An *equational* specification is a sub-equational specification having $\mathrm{E}L := \{(\mathrm{emb}), (\mathrm{comp}), (\mathrm{ref}), (\mathrm{sym}), (\mathrm{trans})\}$ as modes of inferences. A statement (s, t) of such a specification will be called an *equation*, and written as $s \approx t$.

The equational specification $\mathcal{EM}ul$ in the introduction consists of four Σ-equations over X, that is, $\Sigma \cup X$-equations, with Σ as in Example 1.

Example 3. In an *ordering* specification all modes of inference except for (sym) are present $\mathrm{R}L := \{(\mathrm{emb}), (\mathrm{comp}), (\mathrm{ref}), (\mathrm{trans})\}$.[2] A statement (s, t) of such a specification will be called an *ordering*, and written as $s \gtrapprox t$.

The specification of ? in the introduction is an ordering specification.

Similarly term rewriting systems[3] are rendered as sub-equational specifications by taking $\{(\mathrm{emb}), (\mathrm{comp})\}$ as modes of inference. Its statements are written using $\rightarrow$, as usual.[4] The TRS corresponding to $\mathcal{EM}ul$ will be denoted by $\mathcal{RM}ul$.

We will not list all possible sets of inference modes, but only mention one more example, which will be used later.

Example 4. Removing $\{(\mathrm{ref}), (\mathrm{sym})\}$ from the modes of inference of $\mathcal{EM}ul$ yields what we call a *positive* ordering specification $\mathcal{TM}ul$, having as fourth component $\mathrm{T}L := \{(\mathrm{emb}), (\mathrm{comp}), (\mathrm{trans})\}$. A statement (s, t) of such a specification will be called a *positive* ordering, and written as $s > t$.

[2] This corresponds to Meseguer's rewriting logic.

[3] To be precise, our term rewriting systems (TRSs) correspond to the pseudo-TRSs of [1, page 36] since we do not impose the usual further restrictions on rules.

[4] Note that although transition system specifications usually employ the $\rightarrow$-notation as well, the (comp)-inference mode is absent for them.

3 Birkhoff

In this section the correspondence between validity (Subsection 3.1) and derivability (Subsection 3.2) for sub-equational specifications is presented in two stages. In Subsection 3.3 we first present a correspondence between *relational* validity and derivability, which is then extended in Subsection 3.4 to a correspondence between validity and derivability by a quotient construction.

3.1 Validity

As usual, algebras are used to give meaning to the terms of a sub-equational specification. However, the notion of validity of a statement (s, t) with respect to a specification $\mathcal{S}$ will now be parametrised over its modes of inference.

Definition 5. *A* Σ-algebra $\mathcal{A}$ *consists of a* carrier *set* A*, and a mapping that associates with each symbol* $f \in \Sigma^{(n)}$ *a function* $f^{\mathcal{A}}{:}A^n {\to} A$*, for every* n*.*

An *assignment* is an X-algebra. For a Σ-algebra $\mathcal{A}$ and an assignment α having the same carrier, $\mathcal{A} \cup \alpha$ denotes the obvious $\Sigma \cup X$-algebra.

Example 5. 1. The algebra $\mathcal{N}at$ of the introduction is a Σ-algebra, for the signature Σ of Example 1. For the same carrier, α of the introduction is an example of an assignment.
2. The Σ*-term* algebra $\mathcal{T}(\Sigma)$ has $T(\Sigma)$ as carrier, and interpretation defined by, for all n, all $f \in \Sigma^{(n)}$, and all $s_1, \ldots, s_n \in T(\Sigma)$: $f^{\mathcal{T}(\Sigma)}(s_1, \ldots, s_n) := f(s_1, \ldots, s_n)$.

Definition 6. *A* Σ-homomorphism h *from a* Σ*-algebra* $\mathcal{A}$ *to a* Σ*-algebra* $\mathcal{B}$*, is a map from the carrier* A *of* $\mathcal{A}$ *to the carrier* B *of* $\mathcal{B}$*, such that for all* n*, all* $f \in \Sigma^{(n)}$*, and all* $a_1, \ldots, a_n \in A$*:* $h(f^{\mathcal{A}}(a_1, \ldots, a_n)) = f^{\mathcal{B}}(h(a_1), \ldots, h(a_n))$*.*

It is easy to see that $\mathcal{T}(\Sigma)$ is *initial* among Σ-algebras, i.e. for any Σ-algebra $\mathcal{A}$, there is a unique homomorphism from $\mathcal{T}(\Sigma)$ to $\mathcal{A}$, which we denote by $[\![\mathcal{A}]\!]$.

Example 6. 1. The unique homomorphism $[\![\mathcal{N}at \cup \alpha]\!]$ which maps $\mathcal{T}(\Sigma, X)$ to $\mathcal{N}at \cup \alpha$, with $\mathcal{N}at$ and α as in Example 5.1, is concretely defined by:
   - $x \mapsto 2$, for $x \in X$
   - $f(s_1, \ldots, s_n) \mapsto f^{\mathcal{N}at}(n_1, \ldots, n_n)$, for $f \in \Sigma$ and $s_i \mapsto n_i$

   For instance, $\mathtt{M}(\mathtt{S}(x), \mathtt{S}(\mathtt{0}))$ is mapped to $(2 + 1) \times (0 + 1)$, i.e. to 3.
2. A *substitution* is the unique homomorphism $[\![\mathcal{T}(\Sigma, X) \cup \sigma]\!]$ of some assignment σ. For instance, if σ assigns $\mathtt{S}(\mathtt{S}(\mathtt{0}))$ to x, then applying the substitution to $\mathtt{M}(\mathtt{S}(x), \mathtt{S}(\mathtt{0}))$ yields $\mathtt{M}(\mathtt{S}(\mathtt{S}(\mathtt{S}(\mathtt{0}))), \mathtt{S}(\mathtt{0}))$. We will often abbreviate the substitution to just σ.

Definition 7. *Let* $\mathcal{S} := \langle \Sigma, X, S, L \rangle$ *be a specification. A* relational model *of* $\mathcal{S}$ *is pair* $(\mathcal{A}, R)$ *consisting of a* Σ*-algebra* $\mathcal{A}$ *and a relation* R *on the carrier* A *of the algebra, satisfying each rule* ℓ *in Table 2, for* $\ell \in L$*. Here*

$$\frac{}{s\ R\ t}\ (\text{emb},\ (s,t) \in S) \qquad \frac{a_1, \ldots, a_n =[R]\ b_1, \ldots, b_n}{f^{\mathcal{A}}(a_1, \ldots, a_n)\ R\ f^{\mathcal{A}}(b_1, \ldots, b_n)}\ (\text{comp},\ f \in \Sigma)$$

$$\frac{}{a\ R\ a}\ (\text{ref}) \quad \frac{a\ R\ b}{b\ R\ a}\ (\text{sym}) \quad \frac{a\ R\ b \quad b\ R\ c}{a\ R\ c}\ (\text{trans})$$

Table 2. Relational models

- $s\ R\ t$ *expresses that for all assignments* α*, it holds* $[\![\mathcal{A} \cup \alpha]\!](s)\ R\ [\![\mathcal{A} \cup \alpha]\!](t)$.
- $=[R]$ *expresses that corresponding components of* $a_1, \ldots, a_n$ *and* $b_1, \ldots, b_n$ *are identical, except for one index, say* i*, for which* $a_i\ R\ b_i$.

(s,t) *is* relationally valid *in* $\mathcal{S}$*,* $\models s\ \mathcal{S}\ t$*, if* $s\ R\ t$ *holds in every relational model* $(\mathcal{A}, R)$ *of* $\mathcal{S}$*.*

Remark 1. Since $s\ R\ t$ depends on $\mathcal{A}$ as well, formally we should consider it to be an abbreviation of $s\ R_{\mathcal{A}}\ t$. One may think of relational models as models of a predicate logic with one binary predicate symbol.

The (comp)-rule is a direct generalisation of the usual compatibility rules found in mathematics and rewriting. In the following examples, the relational models for equational, ordering, and positive ordering specifications are characterised. To that end, recall that a relation R is a *congruence* relation for an algebra $\mathcal{A}$, if it is an equivalence relation which is *preserved* by the *operations* of $\mathcal{A}$, i.e. such that for every n-ary operation ϕ, if $a_1\ R\ b_1, \ldots, a_n\ R\ b_n$, then $\phi(a_1, \ldots, a_n)\ R\ \phi(b_1, \ldots, b_n)$.[5] In each example, we will assume the relational model to be $(\mathcal{A}, R)$.

Example 7. . In case of an equational specification, R is seen to be a congruence relation as follows. Since $\{(\text{ref}), (\text{sym}), (\text{trans})\} \subseteq \mathrm{E}L$, R is an equivalence relation. To see that it is a congruence relation, suppose ϕ is an n-ary operation in $\mathcal{A}$ and $a_1\ R\ b_1, \ldots a_n\ R\ b_n$, then we conclude from

$$\begin{array}{rl} \phi(a_1, \ldots, a_n) & R\ \phi(b_1, \ldots, a_n) \\ & \vdots \qquad \ddots \\ & R\ \phi(b_1, \ldots, b_n) \end{array}$$

using $\{(\text{comp}), (\text{trans})\} \subseteq \mathrm{E}L$.

Models of equational specifications in the standard sense of the introduction give rise to relational models, just by pairing them up with the identity relation id. For instance, $(\mathcal{N}at, \mathsf{id})$, is a relational model of $\mathcal{EM}ul$. id is trivially a congruence relation, and (emb) is forced to hold by the assumption that $\mathcal{N}at$ is a model, in the standard sense, of $\mathcal{EM}ul$.

[5] Hence the distinction between compatibility and congruence is that the latter requires all corresponding premisses to be related, whereas the former requires exactly one pair of corresponding premisses to be related (and the rest to be identical).

However, note that R is in general *not* forced to be the identity relation. For instance, an example of a relational model for the equational specification $\mathcal{EM}ul$ consists of its term algebra $\mathcal{T}(\Sigma, X)$ and the convertibility relation $\leftrightarrow^*_{\mathcal{EM}ul}$ (see Example 16).

Example 8. In case of an ordering specification, R is seen to be an operation-preserved quasi-order. That it is a quasi-order, i.e. reflexive and transitive, follows since $\{(\text{ref}), (\text{trans})\} \subseteq \mathrm{E}L$. That it is operation-preserved follows as in the previous item.

Example 9. In case of a positive ordering specification, the relation R is an operation-preserved transitive relation.

Example 10. Any relational model for the equational specification $\mathcal{EM}ul$ is is automatically a relational model for its associated TRS $\mathcal{RM}ul$. Of course, this does not hold the other way around. For instance, combining the polynomial interpretation of [1, Example 6.2.13] with the natural order $>$ on the natural numbers yields a relational model of $\mathcal{RM}ul$, but not of $\mathcal{EM}ul$, because $>$ is not symmetric. Although symmetry is lacking, transitivity is not, hence this interpretation *is* a model of the positive ordering specification $\mathcal{TM}ul$.

As the first example shows, there is a mismatch between the notion of a model and that of a relational model. It is analagous to the difference between the notions of model of predicate logic with and without equality: in the former the interpretation of the binary equality predicate is fixed to the identity relation, whereas in the latter its interpretation can in principle be any relation (possibly satisfying some constraints). That is, there are many more relational models than there are models. This mismatch will be overcome in Subsection 3.4.

3.2 Derivability

Definition 8. *The judgment that a statement* (s, t) *is derivable by means of sub-equational logic, for a given sub-equational specification* $\mathcal{S}$*, is denoted by* $\vdash s \; \mathcal{S} \; t$*. The axioms and rules of* sub-equational logic *are the ones listed in Table 3. The* theory $\underline{\mathcal{S}}$ *of* $\mathcal{S}$ *is the relation on terms, consisting of all derivable pairs.*

Here derivability of a statement means that it is the conclusion of some proof tree built from the inference rules, as usual. Note that an inference rule only applies when it is an allowed mode of inference, according to the specification. Furthermore, not all modes of inference of standard equational logic as presented in Table 1 are (directly) at our disposal in sub-equational logic, not even for an equational sub-equational specification, where all modes of inference are available. The reason is that the standard inference rules of equational logic exhibit some dependencies which we have avoided here, in order to make the connexion between syntax and semantics smoother. In particular, the equation- and substitution-rule have been merged into the (emb)-rule. Furthermore, the (comp)-rule allows one to relate only one argument at the time whereas the standard presentation has a congruence-rule. Nevertheless, the two presentations are easily seen to be equivalent as illustrated by the following example.

$$\frac{}{\sigma(s)\ \underline{\mathcal{S}}\ \sigma(t)}\ (\text{emb},\ (s,t)\in S,\ \sigma{:}X\to T(\Sigma,X)) \qquad \frac{s_1,\ldots,s_n =[\underline{\mathcal{S}}]\ t_1,\ldots,t_n}{f(s_1,\ldots,s_n)\ \underline{\mathcal{S}}\ f(t_1,\ldots,t_n)}\ (\text{comp},\ f\in\Sigma)$$

$$\frac{}{s\ \underline{\mathcal{S}}\ s}\ (\text{ref}) \quad \frac{s\ \underline{\mathcal{S}}\ t}{t\ \underline{\mathcal{S}}\ s}\ (\text{sym}) \quad \frac{s\ \underline{\mathcal{S}}\ t \quad t\ \underline{\mathcal{S}}\ u}{s\ \underline{\mathcal{S}}\ u}\ (\text{trans})$$

Table 3. Sub-equational logic for sub-equational specification $\mathcal{S} := \langle \Sigma, X, S, L\rangle$

Example 11. Redrawing the proof tree of the introduction for the sub-equational specification corresponding to $\mathcal{EM}ul$, omitting parentheses and $\underline{\mathcal{EM}ul}$ to save space, yields

$$\dfrac{\dfrac{}{(\mathtt{M}(\mathtt{S}x,\mathtt{S0}),\mathtt{A}(\mathtt{S}x,\mathtt{M}(\mathtt{S}x,\mathtt{0})))}\,(\sigma) \qquad \dfrac{\dfrac{\dfrac{}{(\mathtt{M}(\mathtt{S}x,\mathtt{0}),\mathtt{0})}\,(\sigma)}{(\mathtt{A}(\mathtt{S}x,\mathtt{M}(\mathtt{S}x,\mathtt{0})),\mathtt{A}(\mathtt{S}x,\mathtt{0}))}\,(\text{comp},\ \mathtt{A}) \quad \dfrac{}{(\mathtt{A}(\mathtt{S}x,\mathtt{0}),\mathtt{S}x)}\,(\sigma)}{(\mathtt{A}(\mathtt{S}x,\mathtt{M}(\mathtt{S}x,\mathtt{0})),\mathtt{S}x)}\,(\text{trans})}{(\mathtt{M}(\mathtt{S}x,\mathtt{S0}),\mathtt{S}x)}\,(\text{trans})$$

More precisely, for a given equational specification $\mathcal{E}$, its derivability in equational logic $\mathcal{E} \vdash s \approx t$ coincides with its derivability $\vdash s\ \mathcal{E}\ t$ in equational sub-equational logic.

3.3 Relational term model

Derivability can be related to relational validity, by constructing a so-called relational term model for a specification.

Lemma 1 (Term Model). *$\vdash s\ \mathcal{S}\ t$ iff $\models s\ \mathcal{S}\ t$, for any specification $\mathcal{S}$.*

Proof. Define the relational *term* model $\mathcal{M}(\mathcal{S})$ of a sub-equational specification $\mathcal{S} := \langle \Sigma, X, S, L\rangle$ as the pair $(\mathcal{T}(\Sigma,X), \underline{\mathcal{S}})$, where $\mathcal{T}(\Sigma,X)$ is the term algebra and $\underline{\mathcal{S}}$ the theory of $\mathcal{S}$.

To prove the if-direction (completeness), it suffices to prove that $\mathcal{M}(\mathcal{S})$ is a relational model for $\mathcal{S}$, by the choice of the theory of $\mathcal{S}$ as the relation of $\mathcal{M}(\mathcal{S})$. Since $\mathcal{T}(\Sigma,X)$ is a $\Sigma\cup X$-algebra by Example 5, it certainly is a Σ-algebra. Hence to verify that $\mathcal{M}(\mathcal{S})$ is indeed a relational model for $\mathcal{S}$, it remains to show that rule ℓ holds in theory $\underline{\mathcal{S}}$, for each $\ell \in L$. Intuitively, this will hold by the 1–1 correspondence between the rules of relational models in Table 2 and the inference rules of sub-equational logics in Table 3. For a proof, we distinguish cases for the rules.

(emb) Suppose $(s,t)\in S$. By the (emb)-rule of Table 2, we must verify that for any assignment α, it holds that $[\![\mathcal{T}(\Sigma,X)\cup\alpha]\!](s)$ is related by $\underline{\mathcal{S}}$ to $[\![\mathcal{T}(\Sigma,X)\cup\alpha]\!](t)$. By the definition of substitution, this is just the same as saying that $\sigma(s)$ is related by $\underline{\mathcal{S}}$ to $\sigma(t)$ for any substitution σ. Which holds by the (emb)-inference rule of the logic.

(comp), (ref), (sym), (trans) Each rule in Table 2 directly follows from the corresponding rule of Table 3, where (comp) also uses that symbols are interpreted as themselves in relational term models.

To prove the only-if-direction (soundness), it suffices to prove by induction on derivations (proof trees) that pairs in the theory $\underline{\mathcal{S}}$, are related in any relational model $(\mathcal{A}, R)$ of $\mathcal{S}$. The proof is by cases on the modes of inference in L, showing that the statement holds for a proof whose conclusion uses inference rule ℓ, by using rule ℓ of Table 2.

(emb) Suppose $(s,t) \in S$ and let σ be some subtitution. We have to show $[\![\mathcal{A} \cup \alpha]\!](\sigma(s)) \; R \; [\![\mathcal{A} \cup \alpha]\!](\sigma(t))$, for any assignment α. Suppose we can show the so-called *semantic* substitution lemma:

$$[\![\mathcal{A} \cup \alpha]\!](\sigma(u)) = [\![\mathcal{A} \cup \alpha_\sigma]\!](u) \tag{6}$$

where the assignment α_σ maps a variable $x \in X$ to the value of $\sigma(x)$ in the algebra $\mathcal{A}$ under the assignment α, i.e. to $[\![\mathcal{A} \cup \alpha]\!](\sigma(x))$.
Then we are done, since

$$[\![\mathcal{A} \cup \alpha]\!](\sigma(s)) = [\![\mathcal{A} \cup \alpha_\sigma]\!](s) \; R \; [\![\mathcal{A} \cup \alpha_\sigma]\!](t) = [\![\mathcal{A} \cup \alpha]\!](\sigma(t))$$

by (emb) of Table 2, and the semantic substitution lemma (twice).
It remains to show (6), which is proven by induction on $u \in T(\Sigma, X)$.
(variable)

$$\begin{aligned} [\![\mathcal{A} \cup \alpha]\!](\sigma(x)) &= \alpha_\sigma(x) \\ &= [\![\alpha_\sigma]\!](x) \\ &= [\![\mathcal{A} \cup \alpha_\sigma]\!](x) \end{aligned}$$

(symbol) For all n, all $f \in \Sigma^{(n)}$, and all $s_1,\ldots,s_n \in T(\Sigma, X)$:

$$\begin{aligned} [\![\mathcal{A} \cup \alpha]\!](\sigma(f(s_1,\ldots,s_n))) &= [\![\mathcal{A} \cup \alpha]\!](f(\sigma(s_1),\ldots,\sigma(s_n))) \\ &= f^{\mathcal{A}\cup\alpha}([\![\mathcal{A} \cup \alpha]\!](\sigma(s_1)),\ldots,[\![\mathcal{A} \cup \alpha]\!](\sigma(s_n))) \\ &=_{\mathrm{IH}} f^{\mathcal{A}\cup\alpha}([\![\mathcal{A} \cup \alpha_\sigma]\!](s_1),\ldots,[\![\mathcal{A} \cup \alpha_\sigma]\!](s_n)) \\ &= f^{\mathcal{A}}([\![\mathcal{A} \cup \alpha_\sigma]\!](s_1),\ldots,[\![\mathcal{A} \cup \alpha_\sigma]\!](s_n)) \\ &= f^{\mathcal{A}\cup\alpha_\sigma}([\![\mathcal{A} \cup \alpha_\sigma]\!](s_1),\ldots,[\![\mathcal{A} \cup \alpha_\sigma]\!](s_n)) \\ &= [\![\mathcal{A} \cup \alpha_\sigma]\!](f(s_1,\ldots,s_n)). \end{aligned}$$

Which concludes the proof of the semantic substitution lemma.
(comp), (ref), (sym), (trans) As for the other direction, these are trivial. □

Note that what we have really is a term model, i.e. terms are interpreted as terms (even stronger: as themselves), unlike the standard term models where terms are interpreted as equivalence classes of terms. The latter will be constructed in the following subsection.

3.4 Quotienting out a maximal congruence

To overcome the mismatch between relational models and models observed above, we show that any relational model can be turned into a model, by quotienting out a maximal congruence relation. Quotienting out a congruence relation $\cong$ consists in taking $\cong$-equivalence classes of elements as new elements.

Definition 9. *Let $\mathcal{M} := (\mathcal{A}, R)$ be a relational model of $\mathcal{S} := \langle \Sigma, X, S, L\rangle$ and let $\cong$ be a congruence relation on the carrier A of $\mathcal{A}$. The* quotient *$\mathcal{M}/_{\cong}$ of $\mathcal{M}$ by $\cong$ is the pair $(\mathcal{A}/_{\cong}, R/_{\cong})$ defined by:*

- *The* quotient *algebra $\mathcal{A}/_{\cong}$ of $\mathcal{A}$ by $\cong$ is defined by:*
  - *The carrier $A/_{\simeq}$ of $\mathcal{A}/_{\simeq}$ consists of the $\cong$-equivalence classes $[a]_{\cong}$ for $a \in A$.*
  - *The interpretation of symbols is given by: for all n, for all $f \in \Sigma^{(n)}$, and all $a_1,\ldots,a_n \in A$*

$$f^{\mathcal{A}/\cong}([a_1]_{\cong}, \ldots [a_n]_{\cong}) := [f^{\mathcal{A}}(a_1, \ldots, a_n)]_{\cong}$$

- *The relation $R/_{\cong}$ on the carrier $A/_{\cong}$ of $\mathcal{A}/_{\cong}$, is defined by:*

$$[a]_{\cong}\ R/_{\cong}\ [b]_{\cong} := a \cong ; R ; \cong b, \textit{ where } ; \textit{ denotes relation composition}$$

Neither the definition of the quotient algebra nor of the quotient relation depends on the choice of the representatives, because $\cong$ is a congruence relation. Under some constraints, taking quotients preserves and 'reflects' modelhood.

Lemma 2 (Quotient). *Let $\mathcal{S} := \langle \Sigma, X, S, L\rangle$ be a specification, $\mathcal{M} := (\mathcal{A}, R)$ be a relational model of $\mathcal{S}$, and $\cong$ a congruence relation on the carrier A of $\mathcal{A}$.*

- *If $\cong \subseteq R^*$, then $\mathcal{M}/_{\cong}$ is a relational model of $\mathcal{S}$ again.*
- *If moreover* (trans) $\in L$, *then $[a]_{\cong}\ R/_{\cong}\ [b]_{\cong}$ implies $a\ R\ b$.*

Proof. We first show the first item. Let $\mathcal{S} := \langle \Sigma, X, S, L\rangle$. We must verify for each $\ell \in L$, that if R satisfies the inference rule ℓ for $\mathcal{M}$, then $R/_{\cong}$ does so for $\mathcal{M}/_{\cong}$. Except for the (emb) rule all cases are easy:

(comp) We have to show that $[a_1]_{\cong}, \ldots [a_n]_{\cong} = [R/_{\cong}]\ [b_1]_{\cong}, \ldots [b_n]_{\cong}$ implies $f^{\mathcal{A}/\cong}([a_1]_{\cong}, \ldots [a_n]_{\cong})\ R/_{\cong}\ f^{\mathcal{A}/\cong}([b_1]_{\cong}, \ldots [b_n]_{\cong})$. By the assumption it holds $a_i \cong b_i$, except say for j, for which $a_j \cong ; R ; \cong b_j$. By (comp) for R and congruence of $\cong$, we obtain $f^{\mathcal{A}}(a_1, \ldots, a_n) \cong ; R ; \cong f^{\mathcal{A}}(b_1, \ldots, b_n)$, from which the claim follows by definition of $f^{\mathcal{A}/\cong}$.

(ref) If R is reflexive, then $\cong ; R ; \cong$ is reflexive by the assumption that $\cong$ is a congruence relation hence reflexive, so $R/_{\cong}$ is reflexive as well.

(sym) If R is symmetric, then $\cong ; R ; \cong$ is symmetric by the assumption that $\cong$ is a congruence relation hence symmetric, so $R/_{\cong}$ is symmetric as well.

(trans) If R is transitive, then $\cong ; R ; \cong$ is transitive by the assumption that $\cong$ is contained in the reflexive–transitive closure of R, so $R/_{\cong}$ is transitive as well.

It remains to verify the (emb) rule holds for $R/_{\cong}$ under the assumption that it holds for R. So suppose $(s,t) \in S$. We have to show

$$[\![\mathcal{A}/_{\cong} \cup \beta]\!](s)\ R/_{\cong}\ [\![\mathcal{A}/_{\cong} \cup \beta]\!](t)$$

for any assignment β of $\cong$-equivalence classes of A, to variables. We will show the so-called *syntactic* substitution lemma:

$$[\![\mathcal{A}/_{\cong} \cup \beta]\!](u) = [[\![\mathcal{A} \cup \alpha]\!](u)]_{\cong} \tag{7}$$

for any assignment α 'picking' elements from those classes, i.e. such that α maps each variable x to an element of x^{β}. Then we conclude, by definition of $R/_{\cong}$:

$$[\![\mathcal{A}/_{\cong} \cup \beta]\!](s) = [[\![\mathcal{A} \cup \alpha]\!](s)]_{\cong}\ R/_{\cong}\ [[\![\mathcal{A} \cup \alpha]\!](t)]_{\cong} = [\![\mathcal{A}/_{\cong} \cup \beta]\!](t)$$

using the assumption that $[\![\mathcal{A} \cup \alpha]\!](s)\ R\ [\![\mathcal{A} \cup \alpha]\!](t)$ for any α. It remains to show (7) for all $u \in T(\Sigma, X)$, which we prove by induction on u.

(variable) Since α was assumed to pick elements from β:

$$[\![\mathcal{A}/_{\cong} \cup \beta]\!](x) = x^{\beta} = [x^{\alpha}]_{\cong} = [[\![\mathcal{A} \cup \alpha]\!](x)]_{\cong}.$$

(symbol) For all n, all $f \in \Sigma^{(n)}$, and all $u_1,\ldots,u_n \in T(\Sigma, X)$:

$$\begin{array}{rcl}
[\![\mathcal{A}/_{\cong} \cup \beta]\!](f(u_1,\ldots,u_n)) & = & f^{\mathcal{A}/_{\cong} \cup \beta}([\![\mathcal{A}/_{\cong} \cup \beta]\!](u_1),\ldots,[\![\mathcal{A}/_{\cong} \cup \beta]\!](u_n)) \\
& =_{\mathrm{IH}} & f^{\mathcal{A}/_{\cong} \cup \beta}([[\![\mathcal{A} \cup \alpha]\!](u_1)]_{\cong},\ldots,[[\![\mathcal{A} \cup \alpha]\!](u_n)]_{\cong}) \\
& = & f^{\mathcal{A}/_{\cong}}([[\![\mathcal{A} \cup \alpha]\!](u_1)]_{\cong},\ldots,[[\![\mathcal{A} \cup \alpha]\!](u_n)]_{\cong}) \\
& = & [f^{\mathcal{A}}([\![\mathcal{A} \cup \alpha]\!](u_1),\ldots,[\![\mathcal{A} \cup \alpha]\!](u_n))]_{\cong} \\
& = & [f^{\mathcal{A} \cup \alpha}([\![\mathcal{A} \cup \alpha]\!](u_1),\ldots,[\![\mathcal{A} \cup \alpha]\!](u_n))]_{\cong} \\
& = & [[\![\mathcal{A} \cup \alpha]\!](f(u_1,\ldots,u_n))]_{\cong}.
\end{array}$$

Showing the second item is easy: by definition $[a]_{\cong}\ R/_{\cong}\ [b]_{\cong}$ iff $a \cong ; R ; \cong b$. By the assumption $\cong \subseteq R^*$, this implies $a\ R^* ; R ; R^*\ b$, from which $a\ R\ b$ follows by the assumption (trans) $\in L$. $\square$

Example 12. Consider the relational model $(\mathcal{T}(\Sigma, X), \leftrightarrow^*_{\mathcal{EM}ul})$ of $\mathcal{EM}ul$ of Example 7. Taking for $\cong$ the convertibility relation $\leftrightarrow^*_{\mathcal{EM}ul}$, we see that it satisfies the first condition of Lemma 2, hence that the convertibility relation itself can be quotiented out. As one easily checks this yields a relational model having the classes of convertible terms as elements, and having the identity relation id as relation. Note that the first component of the resulting model, is a model in the sense of the previous section. That is, we have constructed a model from a relational model.

The construction in the example can be generalised in the sense that if the relation of a relational model contains a non-trivial congruence it can be quotiented out. In fact, we take this as the defining property of a model.

Definition 10. *A* model *of $\mathcal{S}$ is a congruence-free relational model. Here, a relational model $(\mathcal{A}, R)$ of a specification $\mathcal{S}$ is* congruence-free *if the reflexive–transitive closure R^* of R contains no congruence relations other than the identity relation* id*. We say (s,t) is* valid*, written $\models s\ \mathcal{S}\ t$, in case $s\ R\ t$ in all models $(\mathcal{A}, R)$ of $\mathcal{S}$.*

Hence, validity is obtained from relational validity by restricting the relational models to models.

Proposition 1. *For any relational model $\mathcal{M} := (\mathcal{A}, R)$, $\mathcal{M}/_{\cong}$ is a model where $\cong$ is a maximal congruence relation $\cong$, such that $\cong \subseteq R^*$.*

Proof. That a maximal congruence relation exists follows from Kuratowski's Lemma, since the union of the congruence relations in a chain is easily seen to be a congruence relation again. That the quotient $\mathcal{M}/_{\cong}$ is a relational model follows from Lemma 2, and that it is congruence-free holds, since otherwise the 'offending' congruence $\cong'$ could have been composed with $\cong$ right away. More precisely, in such a case, defining a to be related to b iff $[a]_{\cong} \cong' [b]_{\cong}$, would have given a congruence relation on A still contained in R^*, but larger than $\cong$ contradicting the latter's maximality. □

Let R be the relation of a relational model for $\mathcal{S}$.

Example 13. As seen above, R itself is the maximal congruence in the case of an equational specification, and models, i.e. congruence-free relational models, are in 1–1 correspondence with the models of the introduction. That is, for an equational specification $\mathcal{E}$, the standard and sub-equational notions of validity coincide.

Generalising the example, one notes that if R is both transitive and operation-preserved, such as is the case for (positive) ordering logic, then the reflexive closure of $R \cap (R^{-1})$ is the largest congruence relation contained in R^*. This maximal congruence just identifies all objects in strongly connected components of R. (Note that if R is terminating, then quotienting does nothing.) Hence models of ordering and positive ordering specifications have *partial* orders (reflexive, transitive and anti-symmetric relations) and *positive* orders (transitive and anti-symmetric relations) respectively, as relations.

Example 14. The models of ordering specifications are better known as $\mathsf{quasi\text{-}models}$ [1, Definition 6.5.30].

By the quotient construction, checking validity on relational models can be restricted to checking validity on models in case of transitive specifications, that is, which have (trans) as mode of inference.

Lemma 3. $\models\!\!\!\models s\ \mathcal{S}\ t$ *iff* $\models s\ \mathcal{S}\ t$*, for transitive specifications $\mathcal{S}$.*

Proof. The only-if-direction holds since models are a special case of relational models. The if-direction follows, since by Proposition 1, any relational model of $\mathcal{S}$ gives rise to a model, in which s and t are related by the assumption $\models s\ \mathcal{S}\ t$, but then s and t were related in the relational model as well, by the second item of the Quotient Lemma 2 using the assumption that $\mathcal{S}$ is transitive. □

Theorem 1 (Birkhoff). *$\vdash s\ \mathcal{S}\ t$ iff $\models s\ \mathcal{S}\ t$, for transitive $\mathcal{S}$.*

Proof. By Lemmas 3 and 1. $\square$

Example 15. For equational specifications this is just Birkhoff's theorem [3].

For ordering specifications, the theorem states the correspondence between validity w.r.t. Zantema's quasi-models and derivability in Meseguer's rewriting logic (using their own terminology), a result originally due to [6].

4 Logicality

We present a uniform method to define convertibility relations for sub-equational logics (Subsection 4.1) and show their logicality [4] (Subsection 4.2), i.e. show that convertibility coincides with derivability for sub-equational specifications.

4.1 Convertibility

Definition 11. *Let $\mathcal{S}$ be a sub-equational specification with modes of inference L. Its* sub-convertibility *relation $\mathcal{S}(\rightarrow)$ is obtained by starting with the empty relation and closing under the inference rule ℓ of sub-equational logic if $\ell \in L$, in the order:* (emb), (comp), (ref), (sym), (trans).

Let $\mathcal{S}$ be a sub-equational specification. Of course, in case of a rewriting specification, having $\{(\text{emb}), (\text{comp})\}$ as modes of inference, $\mathcal{S}(\rightarrow)$ is just the rewrite (step) relation generated by the rules. Other examples are:

Example 16.
1. For an equational specification $\mathcal{S}(\rightarrow)$ is convertibility $\leftrightarrow^*_{\mathcal{S}}$.
2. For an ordering specification $\mathcal{S}(\rightarrow)$ is rewritability/reachability $\rightarrow^*_{\mathcal{S}}$.
3. For a positive ordering specification $\mathcal{S}(\rightarrow)$ is positive reachability $\rightarrow^+_{\mathcal{S}}$.

Further examples one could think of are e.g. head steps ($\{(\text{emb})\}$) for modelling process calculi, *Identity* ($\{(\text{ref})\}$) then $\mathcal{S}(\rightarrow)$ is just syntactic identity, or *Empty* ($\emptyset$) for which $\mathcal{S}(\rightarrow)$ is the empty relation.

4.2 Closure

We prove that derivability coincides with convertibility for a given sub-equational specification. As convertibility is defined as a special case of derivability, i.e. by applying the inference rules in the order as given in Definition 11, it is clearly contained in it. To show the other inclusion it suffices to prove that closing under an inference mode preserves closure under inference modes earlier in the order, since then the generated relation must coincide with derivability as the latter is the *least* relation closed under each inference mode. We illustrate this by means of an example.

Example 17. Suppose the relation R is compatible and we take its symmetric closure yielding $R \cup R^{-1}$. We must show that compatibility is preserved. That is, we must prove that $f(s_1, \ldots, s_n)\ R \cup R^{-1}\ f(t_1, \ldots, t_n)$ holds, under the assumption $s_1, \ldots, s_n =[R \cup R^{-1}]\ t_1, \ldots, t_n$. We distinguish cases according to whether compatibility is due to R or R^{-1} holding between two premisses.

- If compatibility is due to R, then the result follows by (comp) for R.
- If the assumption is due to R^{-1}, then $t_1, \ldots, t_n =[R]\ s_1, \ldots, s_n$, hence by (comp) $f(t_1, \ldots, t_n)\ R\ f(s_1, \ldots, s_n)$, hence by (sym) $f(s_1, \ldots, s_n)\ R \cup R^{-1}$ $f(t_1, \ldots, t_n)$.

Checking preservation for all other combinations is as easy.

Proposition 2. *Closing relations in the order of Definition 11 preserves the properties/inference rules earlier in the order.*

Proof. First, note that all operations are monotonic in the sense that they may generate new conclusions, but preserve all existing ones. As the (emb)and (ref)inference rules have empty premisses, monotonocity explains the corresponding rows in the following table, which displays vertically the property which is to be preserved under closing with respect to the horizontally indicated inference mode.

	(emb)	(comp)	(ref)	(sym)	(trans)
(emb)	x	mon	mon	mon	mon
(comp)	x	x	(ref)	(sym)	(trans)
(ref)	x	x	x	mon	mon
(sym)	x	x	x	x	(trans)
(trans)	x	x	x	x	x

No closures are taken after (trans), which explains the last row. Preservation in the (comp)- and (sym)-rows follows by easy structural manipulations, using the inference rule given in the table in the end. For instance, the proof that (comp) is preserved under (sym) employs (sym) as final rule, as shown in Example 17. The other entries are dealt with in an analogous way. □

Remark 2. Alternatively, one could permute any two consecutive inference rule in a derivation which are in the 'wrong' order. One easily shows that permutation is always possible, that the process terminates (use e.g. recursive path orders), and that the resulting derivation (the normal form) is a conversion.

Theorem 2 (Logicality). *$\vdash s\ \mathcal{S}\ t$ iff $s\ \mathcal{S}(\rightarrow)\ t$, for specifications $\mathcal{S}$.*

Proof. ($\Rightarrow$) It suffices to verify that the term algebra $T(\Sigma, X)$ with relation $\mathcal{S}(\rightarrow)$ constitutes a relational model. It follows directly from Proposition 2.
($\Leftarrow$) Trivial, since $\mathcal{S}(\rightarrow)$ is constructed by successively closing under the inference rules which are also part of the sub-equational specification $\mathcal{S}$. □

As a final application combining the Birkhoff and Logicality theorems consider the following result due to Zantema [1, Theorem 6.2.2]:

Theorem 3. *A TRS is terminating if and only if it admits a compatible well-founded monotone algebra.*

Proof. View the TRS as a positive ordering specifications. From the above we then have that $\rightarrow^+$ is sound and complete w.r.t. positively ordered models. If the order is required to be well-founded such models coincide with compatible well-founded monotone algebras. Hence the if-direction follows from the existence of such a model by soundness. The only-if direction follows by the relational term model construction, and the observation made above that quotienting a terminating relation does nothing.

Note that for this example to work it was necessary to drop (ref), i.e. one could work with neither equational nor rewriting logic. Also, building-in transitivity in the order of a monotone algebra would not have been necessary; working with a terminating relation instead would be fine as well. More generally, often a *big step* semantics can easily be replaced by a *small step* semantics without problems.

5 Conclusion

We have given a uniform presentation of Birkhoff-style sound- and completeness results for various sub-equational logics. Moreover, we have given a uniform proof of logicality of rewriting for each of them. Although the results are not very surprising, we have not seen such a uniform presentation before. Moreover we do think the resulting presentation is elegant and the analysis required and performed is useful.

Acknowledgments Thanks are due to ETL, Tsukuba and the Institute of Information Sciences and Electronics, of Tsububa University for travel support, to the referees for constructive criticism, and to J. Joosten, A. Visser, and the attendants of the ETL-rewrite day, for feedback.

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