Uniform Completeness

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Abstract

We introduce uniform completeness and give a local characterisation of it. We show it yields a complete method for showing completeness of rewrite systems.

1 Introduction

The canonical way to establish completeness of rewrite systems is by showing local confluence and termination, as enabled by Newman's Lemma [4, Theorem 3]. In [5] we provided the following alternative route to establishing completeness:

Corollary 1. *If $\to$ is normalising (WN) and ordered weak Church–Rosser, then $\to$ is complete.*

Proof. Ordered weak Church–Rosser (Definition 3) entails by [5, Theorem 3] that $\to$ has *random descent*,[1] i.e. that any object convertible to normal form reduces to it, and always in the same number of steps. Hence the additional assumption of normalisation entails completeness. □

Observe that this alternative route in fact yields a result stronger than completeness, namely that all reductions from a given object to normal form are not only terminating and end in the same object, but also that the (length) measure of these reductions is always the same.

In this short paper we show the alternative route to be complete, if $\to$ is complete then $\to$ is normalising and ordered weak Church–Rosser, when the order-constraint in the latter is allowed to depend on an arbitrary *measure* [8]. We showcase the technique by some simple examples.

Example 1. *Consider the rewrite system $\to$ having steps $a \to b$, $b \to c$ and $a \to c$. It is trivially complete. Measuring the steps by natural numbers (with addition) as $a \to_1 b$, $b \to_1 c$ and $a \to_2 c$, yields a system that is ordered weak Church–Rosser in the sense of [8] (for this chosen measure). In particular, all reductions from a to its normal form c have the same measure 2. However, $\to$ is not ordered weak Church–Rosser in the sense of [5], i.e. measuring the numbers of steps, since the* length *of the reduction $a \to b \to c$ from a to c is then 2, whereas that of $a \to c$ is 1.*

Note that adjoining a step $c \to c$ in the example yields a system that although no longer WN, still *is* ordered weak Church–Rosser and that for *any* measure, also for the length measure. The following two examples are WN, and ordered weak Church–Rosser for the length measure.

Example 2 ([5, Example 7])**.** *Consider the rewrite system $\to$ that sorts strings of letters by repeatedly swapping adjacent out-of-order letters. It is easy to see that if $s \leftarrow t \to u$, then $s \to^n t \ {}^n\!\!\leftarrow u$, with $n \in \{0,1,2\}$ depending on whether the respective swaps are the same, non-overlapping, or overlapping. For instance, for $bca \leftarrow cba \to cab$ we have $bca \to bac \to abc \leftarrow acb \leftarrow cab$. Therefore $\to$ is ordered weak-Church–Rosser. As $\to$ is normalising, e.g. by bubble-sort, it terminates uniquely by Corollary 1, taking the same number of steps given a string.*

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[1] So named in [5] to honour [4]. The notion was given a *description but not a definition* [9] by Newman.

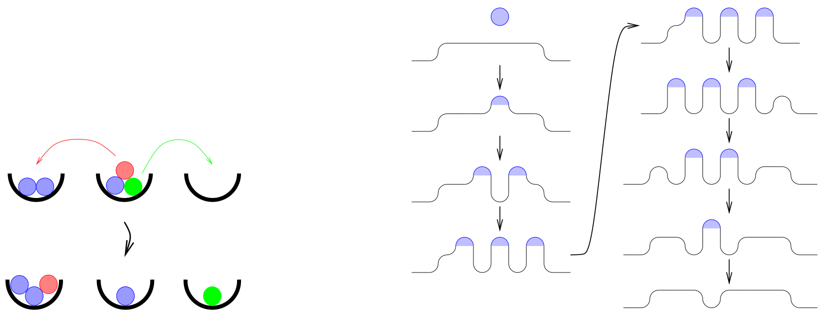

Figure 1: Bowls and beans move (left) and normalisation (right)

Example 3 (cf. [2, Example 8.3]). *Consider the rewrite system* $\to$ *that given a two-sided infinite sequence* $\mathbb{Z} \to \mathbb{N}$ *of* bowls *each holding a number of* beans *may, for a bowl holding at least two beans, move one bean to each adjacent bowl, see Figure 1 left. Formally,* $t \to s$ *if* s *is the same as* t *except that for some* $z \in \mathbb{Z}$ *such that* $t(z) \geq 2$*, we have* $s(z-1) = t(z-1)+1$*,* $s(z) = t(z)-2$ *and* $s(z+1) = t(z+1)+1$*. It is easy to see that if* $s \leftarrow t \to u$*, then* $s \to^n t \,{}^n\!\leftarrow u$*, with* $n \in \{0,1\}$ *depending on whether the respective moves are the same or not. Therefore* $\to$ *is ordered weak-Church–Rosser. That* $\to$ *is normalising follows e.g. by noting that a sequence is in normal form iff each bowl holds at most one bean, and that adding a single bean to a normal form produces 'waves' that first extend outwardly and then, after reaching their limit, die down inwardly, see Figure 1 right. By Corollary 1* $\to$ *is complete and by the observation, given a sequence normalising it always takes the same number of steps before reaching the normal form.*

2 Uniformly complete ⇔ has peak random descent

For a property Π of objects of a rewrite system its restriction to *meaningful* objects is of interest, with the crudest approximation of meaningful being that objects be convertible to normal form.

Definition 1. *A rewrite system* $\to$ *is* uniformly Π *if* $\Pi(a)$ *for all* a *with* $a \leftrightarrow^* \cdot \not\to$*.*

Uniformity resides in there being no steps between the objects that have property Π and those that do not. Obviously, since *uniform* Π*-ety* requires Π to hold only for the objects convertible to normal form, it is in general weaker than Π. In the literature *uniform termination* has been studied, e.g. in [3] (cf. Remark 1). Here we introduce and study *uniform completeness*, i.e. uniform Π-ety for Π the property defined by $\Pi(a) \coloneqq$ *a is confluent and terminating*. We relate uniform completeness to extant notions from rewriting [3, 10].

Proposition 1. $\to$ *is uniformly complete iff* $\to$ *both is uniformly terminating and has NF.*

Proof. For the only–if-direction, suppose $\to$ is uniformly complete. If an object is normalising, it is convertible to some normal form hence terminating by assumption. If an object is convertible to any normal form it is confluent by assumption so reduces to it, i.e. it has NF (the normal form property [10, Definition 1.1,13(iv)]). For the if-direction, suppose $\to$ is uniformly terminating and has NF. Then if an object is convertible to normal form, it is terminating by uniform termination, and the reduction must end in the normal form by NF. □

Example 4. *Since* β*-reduction is confluent it has NF Combined with uniform termination of the* λI*-calculus [1, p.20 7XXV] and of the simply typed* λ*-calculus it yields uniform completeness of both. The untyped* λ*-calculus is not uniformly complete; cf.* $(\lambda x.y)\,((\lambda z.z\,z)\,(\lambda u.u\,u))$*.*

Remark 1. *For* reduction-closed *properties* Π*, i.e.* if $\Pi(a)$ and $a \to b$ then $\Pi(b)$*, such as termination, a rewrite system being uniformly* Π *is equivalent to the absence of so-called* Π-critical *steps, steps from an object not having property* Π *to one having it. In fact, that characterisation was used as the* definition *of uniform termination in [3, Definition 2].*[2] *Absence of* Π*-critical steps can be positively stated as that all steps are* Π-perpetual*, i.e. preserve* $\neg\Pi$*, cf. [3].*

The key result of this short note is a *local* [4] characterisation of uniform completeness via the notion of peak random descent as introduced in [8, Definition 22]. Peak random descent expresses that for any peak of reductions where the first ends in a normal form, the second can be extended by a reduction to the same normal form such that both resulting legs have the same measure, for a given measure on steps. Here, each step is measured by assigning to it some element of a monoid, distinct from the unit. This is then naturally extended to (finite) reductions by using the unit and operation of the monoid from tail to head, e.g. the measure of $\to_1 \cdot \to_2$ in the monoid of natural numbers with zero and addition is $0+2+1=3$. In order to formalise the notion of peak random descent, we equip rewrite systems with such a measure [8].

Definition 2. *A monoid with addition* $+$ *and zero* $\bot$ *is a* derivation *monoid if it comes equipped with a well-founded partial order* $\leq$ *such that* $\bot$ *is least and* $+$ *is monotonic in both arguments and strictly so in its second. A* measure *for a rewrite system is a map from steps to the non-*$\bot$*-elements of a derivation monoid. The measure of a finite reduction is the sum of the measures of the steps in it, from tail to head. This is extended to infinite reductions by representing these as* steps *of the rewrite system* $\to^\infty$ *[8, Definition 10] having a step from a to b for* any *infinite* $\to$*-reduction from a and* any *b. Such* $\to^\infty$*-steps are measured by* $\top$ *with* $\top$ *added as a top to the monoid.*[3] *That allows to represent reductions that may be either finite or infinite by* $\to^{\circledast} := (\to \cup \to^\infty)^*$*, called* extended *reduction in [8].*

We use $\mu, \nu, \ldots$ to denote arbitrary measures and $m, n, \ldots$ to denote *finite* ($\neq \top$) ones.

Example 5.

- *The ordinals equipped with zero* 0*, ordinal addition* $+$ *and the standard order* $\leq$ *on them constitute a derivation monoid. Note* $+$ *is not strictly monotonic in its first argument, e.g.* 0 *is smaller than* 1 *but* $0+\omega = \omega = 1+\omega$*.*
- *The* length *measure is obtained by assigning the ordinal* 1 *to all steps. An infinite reduction will then have measure* $\top$*, not* ω*. This is because to represent an infinite reduction, we need to employ at least one* $\to^\infty$*-step, making the whole reduction have measure* $\top$*. (Note the measure is independent of 'what infinite part' is represented by a* $\to^\infty$*-step.)*

The above slightly generalises [8, Definition 4] motivated by the desire and need to use ordinal measures. We assume $+$ to be strictly monotonic only in its 2nd argument in a derivation monoid, whereas in [8] strict monotonicity in both arguments was assumed. Moreover, we measure reductions from tail to head whereas in [8] they were measured from head to tail. The latter difference is only apparent as one can always transform to the other direction by using $\lambda xy.y + x$ instead of $+$. The reason for changing it nonetheless is that it allows to keep the standard ordinal operations; cf. Example 5. With this, things carry over *verbatim*:

Definition 3 ([8])**.** $\to$ *has* peak random descent *if* $a\ {}_n^{*}\!\!\leftarrow \cdot \to_\mu^{\circledast} b$ *with a in normal form implies* $a\ {}_{n'}^{*}\!\!\leftarrow b$ *with* $n' + \mu = n$*,*[4] *and* $\to$ *is* ordered *locally confluent (or* ordered *weak Church–Rosser; OWCR), if* $a\ {}_n\!\!\leftarrow \cdot \to_m b$ *implies* $a \to_{\mu'}^{\circledast} \cdot\ {}_{n'}^{*}\!\!\leftarrow b$ *with* $n' + m \leq \mu' + n$*, for some derivation monoid.*[5]

[2] There it is called *uniform normalisation*, which in hindsight seems not the most uniform way of naming.
[3] By adjoining $\top$ to a derivation monoid it is no longer strict in its second argument.
[4] The condition $n' + \mu = n$ implicitly captures that μ be finite, i.e. implicitly excludes infinite right legs.
[5] OWCR $\not\Rightarrow$ WCR. E.g. $b \leftarrow b \leftarrow a \to c \to c$ is OWCR since both $b \to_\top^\infty c$ and $c \to_\top^\infty b$ as b and c are looping.

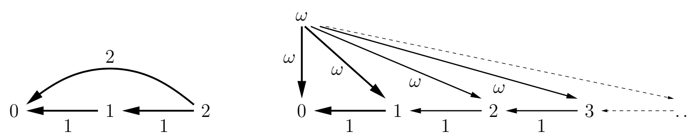

Figure 2: Measures obtained for rewrite systems of Example 1 (left) and Example 6.3 (right)

In a directed acyclic graph weights can be assigned to edges such that all paths from one node to another in it have the same weight, by topological sorting. This key idea of the proof of Lemma 1 (and thereby of this note) is illustrated in Example 6 and Figure 2.

Lemma 1. $\to$ *is uniformly complete iff* $\to$ *has peak random descent.*

Proof. For the only–if-direction first note that the assumption implies that all objects convertible to some normal form are complete. Thus $\to^{+}$ is a well-founded order on them, and we will exploit it to define a measure on the steps in conversions to normal form. We measure steps in the derivation monoid of the ordinals with 0 and ordinal addition $+$, extended with a top $\top$.

We construct measuring functions both for steps and objects, with the measure of an object being based on the measures of all its reductions to normal form. We first partition the objects into those that are convertible to some normal form, and those that are not. We measure the latter, and steps between them, arbitrarily, by 1. An object a of the former is measured by the supremum of the successors of the measures of all b such that $a \to b$. This is well-defined by well-foundedness of $\to^{+}$. In turn, each such step $a \to b$ is measured by the ordinal γ such that $\beta + \gamma = \alpha$, where α, β are the measures of a, b. γ exists and is non-0 per construction.

We claim peak random descent then holds. Note that it suffices to verify it for *finite* peaks $a \;{}_{\alpha}^{*}\!\!\leftarrow \cdot \to_{\beta}^{*} b$ with a in normal form; a right leg containing an $\to^{\infty}$-step would contradict uniform termination. We prove that then $\alpha = \gamma + \beta$ with γ the measure of b, by induction on the length of the peak, distinguishing cases on the direction of its last step. For the empty peak, it is trivial as normal forms have measure 0. Otherwise, $a \;{}_{\alpha'}^{*}\!\!\leftarrow \cdot \to_{\beta'}^{*} b' \leftrightarrow b$, where $\alpha' = \gamma' + \beta'$ holds with γ' the measure of b' by the induction hypothesis. Let δ be the measure of $b' \leftrightarrow b$. If $b' \to b$, then $\alpha = \alpha' = \gamma' + \beta' =^{(\dagger)} (\gamma + \delta) + \beta' = \gamma + (\delta + \beta') = \gamma + \beta$ where $(\dagger)$ holds since γ and γ' are the measures of b and b' and δ the measure of $b' \to b$ so $\gamma' = \gamma + \delta$. If $b' \leftarrow b$, then $\alpha = \alpha' + \delta = (\gamma' + \beta') + \delta = (\gamma' + \beta) + \delta =^{(*)} (\gamma' + \delta) + \beta = \gamma + \beta$, using for $(*)$ that this case can only happen when $\beta = 0 = \beta'$ (while constructing the left leg of the peak, its right leg is empty).

For the if-direction it suffices by Proposition 1 to show uniform termination and the normal form property. To show the former it suffices by Remark 1 to note that if $a \to b$ and b is terminating, the (finite) measure of the step and the reduction to normal form is, by peak random descent, an upper bound on the measures of the reductions from a. The latter follows by induction on the number of peaks in the conversion to normal form, cf. [8, p. 32:3]. □

Example 6. *1. The measures for Example 1 are displayed on the left in Figure 2. Since c is a normal form its measure is* 0*. Since c is the only single-step reduct of b, the measure of b is the successor of the measure of c, i.e.* 1*, so the step* $b \to c$ *has measure* 1 *as well. Finally, both c and b being single-step reducts of a, the measure of a is the supremum of* $\{0 + 1, 1 + 1\}$*, i.e.* 2*, so* $a \to b$ *has measure* 1 *and* $a \to c$ *measure* 2*;*

2. *It is easy to see that in Example 2 objects are measured by their number of* inversions *and that all steps have measure* 1*, e.g. cba has* 3 *inversions and indeed requires* 3 *steps to sort;*

3. *The measures for the rewrite system having steps $a \to b_i$ and $b_{i+1} \to b_i$ for $i \in \mathbb{N}$, are displayed on the right in Figure 2. Proceeding as in the previous item, the only interesting thing to note is that the system is not* finitely branching *(FB). Accordingly a has measure the supremum of $\{i+1 \mid i \in \mathbb{N}\}$, i.e. ω, which thus is the measure of each step from it too.*

Remark 2. 1. *The proof of Lemma 1 uses that for peaks $\beta = 0 = \beta'$ in the case of $b' \leftarrow b$. For arbitrary conversions instead of peaks, this need not hold, and indeed the construction breaks down. To see this, consider the rewrite system $>$ on the ordinals up to and including ω. Proceeding as in the proof of the lemma, the objects and steps of the conversion $0 < 1 > 0 < \omega$ are measured as $0_0 \; {}_1{<} \; 1_1 >_1 0_0 <_\omega \omega_\omega$. But the measure of the backward steps in the conversion is $1 + \omega = \omega$ which is different from the sum $\omega + 1$ of the measures ω of the object ω and 1 of the forward step(s) in the conversion.*

2. *Ordinals serve to deal with systems as in Example 6.3 that are* not *finitely branching. For systems that* are *finitely branching (FB), the natural numbers suffice in the proof of Lemma 1 since then the* supremum *is the* maximum. *For* commutative conversion *monoids [8] such as the natural numbers with zero and addition, the proof generalises from peaks to conversions since then $(*)$ in the proof holds unrestrictedly.*

3 Local characterisation of uniform completeness

Lemma 2. *$\to$ has peak random descent iff $\to$ is ordered locally confluent.*

Proof. First, observe that $\to$ is ordered locally confluent iff it is *ordered* confluent, i.e. $a \; {}_n^*{\leftarrow} \cdot \to_\mu^\circledast b$ *implies* $a \to_{\mu'}^\circledast \cdot \; {}_{n'}^*{\leftarrow} \; b$ *with* $n' + \mu \le \mu' + n$ [8, Definition 12]. This follows from (the proof of) [8, Lemma 18], instantiating both rewrite systems with $\to$, and noting that strictness of monotonicty of $+$ was only used in its second argument (to get, for our conventions, $k' < k' + m_1$).

We show $\to$ is ordered confluent iff it has peak random descent. For the only–if-direction, ordered confluence for $a \; {}_n^*{\leftarrow} \cdot \to_\mu^\circledast b$ with a in normal form implies $a \; {}_{n'}^*{\leftarrow} \; b$ with $n' + \mu \le n$, since a only allows the empty reduction with measure $\bot$. Applying ordered confluence to the converse of the resulting peak, comprising two reductions to normal form a, yields conversely that $n \le n' + \mu$, hence $n' + \mu = n$. For the if-direction we distinguish cases on whether a in the peak $a \; {}_n^*{\leftarrow} \cdot \to_\mu^\circledast b$ is normalising or not. If it is, say $a \to_{m'}^* a'$ with a' in normal form, then by peak random descent for $a' \; {}_{m'+n}^*{\leftarrow} \cdot \to_\mu^\circledast b$ we have $a' \; {}_{n'}^*{\leftarrow} \; b$ with $m' + n = n' + \mu$, as desired. Otherwise, we conclude from $a \to_\top^\infty b$. ☐

Remark 3. *Whereas the first part of the proof of Lemma 2 is a special case of a commutation lemma [8, Lemma 18], the second half is not; that explicitly exploits that extending a reduction by further steps yields such a reduction again; this fails in the commutation case.*

By the two lemmata we conclude to our main result and method to establish completeness.

Theorem 1. *$\to$ is uniformly complete iff $\to$ is ordered locally confluent.*

Corollary 2. *$\to$ is complete iff $\to$ is ordered locally confluent and normalising.*

Example 7. *Consider the term rewrite rule for associativity:*[6] $\varrho(x,y,z) : xyz \to x(yz)$. *It is linear and it has a single critical peak which may be completed into a local confluence diagram with legs $xyzw \to x(yz)w \to x(yzw) \to x(y(zw))$ and $xyzw \to xy(zw) \to x(y(zw))$. To show*

[6] In applicative notation, using association to the left for the implicit infix application symbol @.

OWCR, observe the length measure does not work as the legs have different lengths. Measuring a step contracting $\varrho(t,s,r)$ by twice the number of leaves of t does: both legs then have the same measure: $2n+2n+2m = 2(n+m)+2n$ with n,m the number of leaves of t, s. For non-overlapping peaks ordered local confluence follows from that counting the number of leaves in a term yields a model, i.e. is invariant under ϱ. Since the bullet function of [7, Definition 32] induces a normalising strategy [7, Lemma 35(Extensive)] $\to$ is complete by the corollary.

An algebraic way of defining the measure may be obtained by employing proof terms [10, Chapter 8] to represent reductions resulting, e.g., in representing the legs of the diagram as $\varrho(x,y,z)w \cdot (\varrho(x,yz,w) \cdot x\varrho(y,z,w))$ and $\varrho(xy,z,w) \cdot \varrho(x,y,zw)$. Then the measure is defined by a 2-algebra*, i.e. an algebra for proof terms, building on a 1-algebra, i.e. an algebra on terms. In the 1-algebra, computing the number of leaves, we interpret variables by assigning 1 to them and interpret @ as addition. The 2-algebra, computing the sum of the numbers of leaves in the first argument of each ϱ-redex contracted in a reduction, builds on that by interpreting variables as 0, @ and $\cdot$ as addition, and ϱ as the 1-value of its first argument (a term).*

Example 8. *The rewrite systems in [6, Example 8] are trivially WN and locally Dyck [8, Definition 16] for the length measure, hence OWCR by [8, Theorem 19], so complete by Corollary 2. (Local Dyckness* always *works for complete rewrite systems that are FB; cf. Remark 2.2.)*

4 Conclusion

We have given an alternative complete method for establishing completeness. As also the classical method is complete it's a matter of taste and tool-support which one one prefers. It should be interesting to find a (direct) measure suitable for simply typed $\lambda\beta$ (cf. Example 4).

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