The Z-property and ω-confluence by context-sensitive termination (2nd draft)

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Abstract

We present a method to derive the Z-property for a first-order term rewrite system $\mathcal{T}$ from completeness of an associated context-sensitive term rewrite system $\mathcal{T},\mu$ with replacement map μ. By only requiring left-linearity of $\mathcal{T}$ and that $\mathcal{T}$-critical peaks are also $\mathcal{T},\mu$-critical peaks, we generalise results in the literature. In particular we allow completeness of $\mathcal{T},\mu$ to be established in arbitrary ways, not necessarily by means level-decreasingness or variations thereof as usually assumed. We answer the first of two open problems raised by Gramlich and Lucas in 2006, whether level-decreasingness can be dropped from their preservation of confluence result, in the affirmative, partially. We moreover answer their second open problem, asking whether confluence in the limit holds under mild assumptions, in the affirmative. We consider both the potentially and actually infinite cases, of infinite reductions on finite terms respectively of strongly convergent reductions from finite to (possibly) infinite terms.

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1 Leitmotiv

There is a tight connection between CSR and modularity (starting with Toyama [26]) and neededness (starting with Huet and Lévy [13]). Our overarching Leitmotiv here will be that *layers* relate to *layered terms* as *function symbols* relate to *terms*. This Leitmotiv is at the basis of the categorical (monadic) approach to (modularity in) rewriting introduced by Lüth and Ghani [18], and also of our algebraic approach in [1].[1] Despite that this correspondence has been noted and used before, cf. the introduction of [16] or [5], we think still more leverage (both conceptually and technically) can be gotten out of these approaches, in each of the areas, to prove analogous results not by analogy (redoing; as is currently mostly the case) but by developing and building on a common substratum; the axiomatic needed normalisation results with respect to a general set of results, e.g. (weak) head-normal forms, come to mind [7, 19], cf. [25, Remark 9.2.12].[2]

[1] See in particular Section 6 of [1], where explicit maps between layers (called *components* there) and function symbols (of the component algebra) are set-up.

[2] E.g. a large part of the modularity literature for TRSs is essentially based on that the *rank* (the number of layers) of terms does not increase along rewrite steps. Viewing layers as function symbols, the rank corresponds to the height of a term, so it should be fruitful to factor those modularity results through results on *non-height-increasing* TRSs. (We have obtained preliminary results on this some years ago.)

For the results about about ω-confluence in Section 3, there is a second Leitmotiv, namely that the *active* and *frozen* arguments as determined by the replacement map of a function symbol correspond to *inductive* and *co-inductive* arguments, in a setting combining them both. E.g. infinite terms should then be obtained by a metric completion where the former are at depth 0 and the latter at depth 1.

The above only serves as a backdrop here; we leave its formal development to future research. Our remarks below often serve to shed further light on potential links between the themes mentioned above (modularity, neededness, co-induction), the developments in this note, and the literature on CSR [15, 16, 17],[3] suggesting commonalities and abstractions. Readers not interested in that are advised to skip such remarks.

We base ourselves on [8, 17]. Given a context-sensitive term rewrite system (CSR) $\mathcal{T},\mu$, with $\mathcal{T}$ a term rewrite system (TRS) and μ a replacement map μ, we use $\to$ to denote the rewrite system induced by $\mathcal{T}$, and $\hookrightarrow$ to denote the rewrite system induced by $\mathcal{T},\mu$. Inspired by [10][4] we are interested in methods to *transfer* confluence of a terminating CSR $\mathcal{T},\mu$ to the TRS $\mathcal{T}$, i.e. of $\hookrightarrow$ to $\to$. In Section 2 we derive confluence of the TRS by establishing the stronger Z-property [23] for $\to$ for the so-called layered bullet map $\odot$ that inside–out and layer-wise $\hookrightarrow$-normalises a term. In Section 3 we derive ω-confluence of the TRS via the ω-angle property, cf. [23], for the bullet map $\circ$ mapping a term to its (possibly infinite) normal form via outside–in context-free $\hookrightarrow$-reduction.

▶ Remark 1. A CSR $\mathcal{T},\mu$ is a special case of a context-sensitive conditional expression reduction system in the sense of [14]: it is *unconditional* (no conditions on the substitution in a rewrite step) and its context-sensitivity is *convective*: the restriction on the context in a rewrite step is brought about (only) by the symbols along the *path* to the hole, in a compositional way via the replacement map μ, not by the rest of the context.

2 The Z-property via context-sensitive completeness

In this section we are interested in transferring confluence of $\hookrightarrow$ to that of $\to$. To that end, we will work under the following two assumptions, unless stated otherwise:

(i) $\mathcal{T}$ critical peaks are $\mathcal{T},\mu$ critical peaks.

(ii) $\mathcal{T},\mu$ is a left-linear and complete (confluent and terminating) CSR.

Observe that if the replacement map μ of a CSR $\mathcal{T},\mu$ is *canonical* [8], i.e. if only the variables may occur frozen in the left-hand sides of rewrite rules,[5] then $\mathcal{T},\mu$ satisfies assumption **(i)**.

To *maximise* the chance that the context-sensitive rewrite system $\hookrightarrow$ is terminating, it is best to *minimise* the number of accessible arguments or, stated differently, to *maximise* the number of frozen arguments [8]. That is, letting μ map each function symbol to the empty set $\emptyset$ would be best, but that may not be possible as assumption **(i)** forces for every rule $\ell \to r$ that for every position p in ℓ such that $\ell|_p$ unifies with some left-hand side of a rule, p be accessible / not frozen. Formally, we define a replacement map μ to be *convective* if $\mu^{con} \subseteq \mu$, i.e. if μ is not more restrictive than μ^{con}, where μ^{con} is the most restrictive replacement map such that $i \in \mu^{con}(\ell(q))$ for any $qi \preceq p$ (and that for all such p), guaranteeing that if two

[3] I am not an expert on CSR, so would be interested in being notified of results relevant to the developments here.

[4] In particular in its contemplation of *cofinal* strategies, which raised the obvious question whether the Z-property could play a rôle here, as that gives rise to a (hyper-)cofinal *bullet* strategy [23].

[5] Formally, μ is *canonical* if $\mu^{can} \subseteq \mu$, i.e. if μ is not more *restrictive* than μ^{can}, where μ^{can} is defined by $i \in \mu^{can}(f)$ if for some position p and some rule $\ell \to r$, we have $\ell(p) = f$ and $\ell(pi)$ is a function symbol.

left-hand sides have overlap the one is accessible iff the other is, but nothing more.[6] Our methods will only apply to convective replacement maps.

▶ Remark 2. **(a)** Without assumption **(i)** one can't expect to *transfer* confluence from $\hookrightarrow$ to $\rightarrow$, simply because context-sensitive rewriting in $\mathcal{T}, \mu$ may miss out on (say nothing about) critical peaks of $\mathcal{T}$. For instance, consider the TRS $\mathcal{T}$ with rules $a \rightarrow b$ and $f(\overline{a}) \rightarrow c$ where we used (as we will do below) overlining[7] to indicate that the argument of f is frozen, i.e. that $\mu(f) := \emptyset$. Then $\hookrightarrow$ is confluent, which may be shown by checking that the only $\hookrightarrow$-reducible terms are a and $f(\overline{a})$, and those are deterministic. In particular, we do *not* have $f(\overline{a}) \hookrightarrow f(\overline{b})$ since a is frozen in $f(\overline{a})$, see [8, 17]. However, $\rightarrow$ is not confluent due to the non-joinable critical peak $f(\overline{b}) \leftarrow f(\overline{a}) \hookrightarrow c$.

(b) Neither assumption **(i)** nor assumption **(ii)** is necessary. That assumption **(i)** is not, may be shown by adjoining $c \rightarrow f(\overline{b})$ to $\mathcal{T}$. That preserves confluence of $\hookrightarrow$, which may be transferred to confluence of $\rightarrow$ using that the source of $f(\overline{a}) \rightarrow f(\overline{b})$ is $\hookrightarrow$-reducible to its target: $f(\overline{a}) \hookrightarrow c \hookrightarrow f(\overline{b})$, showing that the problematic critical peak is *redundant*, cf. [11]. We defer the study of redundancy to later work.

(c) In general we have $\mu^{con} \subseteq \mu^{can}$, and this inclusion may be proper. For instance, for *orthogonal* TRSs μ^{con} freezes *all* arguments, $\mu^{con} = \emptyset$, and then our assumptions reduce to that rewriting be root-terminating. That is, up to the root-termination condition, our method below recovers the classical result that orthogonal TRSs are confluent.[8]

▶ **Lemma 3.** *If $t \rightarrow s$ then $t^\bullet \twoheadrightarrow s^\bullet$, where $\bullet$ maps a term to its unique $\hookrightarrow$-normal form.*

Proof. We claim $t \mathrel{\mathrlap{\,\shortparallel}\rightarrow} s$ entails $t^\bullet \twoheadrightarrow \hat{s} \mathrel{\hookleftarrow\!\!\!\!\!\!\twoheadleftarrow} s$ for some $\hat{s}$. From the claim we conclude using $\hat{s} \twoheadrightarrow s^\bullet$ by assumption **(ii)** and $\hookrightarrow \subseteq \rightarrow$. We prove the claim by induction on t w.r.t. $\hookleftarrow$ well-founded by assumption **(ii)**, and by distinguishing cases on $t \mathrel{\mathrlap{\,\shortparallel}\rightarrow} s$:

If $t \mathrel{\mathrlap{\,\shortparallel}\rightarrow} s$ decomposes as $t \hookrightarrow t' \mathrel{\mathrlap{\,\shortparallel}\rightarrow} s$, we conclude by the IH for $t' \mathrel{\mathrlap{\,\shortparallel}\rightarrow} s$ and $t^\bullet = t'^\bullet$.

Otherwise $t \mathrel{\mathrlap{\,\shortparallel}\rightarrow} s$ only contracts non-μ-redexes, occurring at depths at least 1 in t. By assumption **(i)** those cannot have overlap with any redex-pattern at depth 0 in t, as that would give rise to a critical peak of $\mathcal{T}$ that is not a critical peak of $\mathcal{T}, \mu$.

If $t = t^\bullet$ we may trivially set $\hat{s} := s$.

Otherwise, for some t' there is a step $t \hookrightarrow t'$ orthogonal to $t \mathrel{\mathrlap{\,\shortparallel}\rightarrow} s$, hence by the assumed left-linearity of $\mathcal{T}$ the steps commute. Because $t \hookrightarrow t'$ is not below (any redex-pattern in) $t \mathrel{\mathrlap{\,\shortparallel}\rightarrow} s$, the residual of the former after the latter is again a (single) $\hookrightarrow$-step, inducing a diagram of shape $t \hookrightarrow t' \mathrel{\mathrlap{\,\shortparallel}\rightarrow} s' \hookleftarrow s$. By the IH for $t' \mathrel{\mathrlap{\,\shortparallel}\rightarrow} s'$ and assumption **(ii)** we conclude to $t^\bullet = t'^\bullet \twoheadrightarrow \hat{s} \mathrel{\hookleftarrow\!\!\!\!\!\!\twoheadleftarrow} s' \hookleftarrow s$ for some $\hat{s}$, as desired. ◀

▶ Remark 4. **(a)** if $\mathcal{T}, \mu$ is *level-decreasing* [8], i.e. if the depth of each variable occurrence in the right-hand side r of a rule $\ell \rightarrow r$ does not exceed the depth of any of its occurrences (unique in case of left-linearity) in the left-hand side ℓ, then the maximal depth of the steps in $t^\bullet \twoheadrightarrow s^\bullet$ is bounded by the maximal depth of the steps in $t \twoheadrightarrow s$, as seen by enriching the statement and proof with the corresponding invariant; level-decreasingness is then (only) needed in the proof to show that the depth of the residual of a non-μ-step ϕ after a μ-step is bounded by the depth of ϕ.

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[6] The idea of the terminology is to view a term as a fluid, and the paths from the root of a left-hand side to the roots of overlapping left-hand sides as representing flows within the fluid, with the flow enabling *activation* of the latter. A term is in $\hookrightarrow$-normal form iff there's no flow from the root of the term to any redex-pattern, that is, if no redex-pattern can be activated, and it then makes some intuitive sense to speak of its layer at depth 0 as being *solid*.

[7] The overlining notation suggests that the overlined argument is *cut off* from its context, i.e. *frozen*.

[8] It should be interesting to know the frequency of root-termination among orthogonal TRSs (in practice).

(b) If μ is canonical and $\mathcal{T}$ left-linear in a CSR $\mathcal{T},\mu$ then the set of terms in $\hookrightarrow$-normal form constitutes a rather well-behaved set of results, cf. [7, 19] as discussed in Section 1:

- Terms in $\hookrightarrow$-normal form are *preserved under non-μ-reduction*, i.e. for any step $t \to_{\not\mu} s$, if t is in $\hookrightarrow$-normal form then so is s, since each occurrence of the redex-pattern of the left-hand side of a rule must be encompassed by a single layer, so no non-μ-step can contribute to the *creation* of a redex-pattern in a layer closer to the root.
  Both canonicity and left-linearity are necessary. Without left-linearity, a balancing step in some layer may create a redex in a layer closer to the root: $f(\overline{a},b) \to f(\overline{b},b) \hookrightarrow \ldots$ in the CSR with rules $a \to b$ and $f(\overline{x},x) \to \ldots$. Without canonicity redex-patterns may extend over several layers, so may be created by non-μ-steps as witnessed by $f(\overline{a}) \to f(\overline{b}) \hookrightarrow c$ for the (convective) CSR with rules $a \to b$ and $f(\overline{b}) \to c$.
- Terms in $\hookrightarrow$-normal form are *preserved under non-μ-expansion*[9] i.e. for any step $t \to_{\not\mu} s$, if s is in $\hookrightarrow$-normal form then so is t. This holds by the same token as in the previous item, since a step being a μ-step or not is *positional* in that it is exclusively determined by the path to the root of its redex-pattern (cf. [25, Remark 9.3.20]): since the redex-pattern of the lhs of a rule for a μ-step from t must be encompassed by the layer at depth 0 by canonicity, it cannot be *eliminated* (cf. [25, Proposition 9.2.2]) by $t \to_{\not\mu} s$ as that step is in a layer at depth ≥ 1.
  Both left-linearity and canonicity are seen to be necessary as in the previous item, for the same reason; consider $\ldots \hookleftarrow f(\overline{a},\overline{a}) \to f(\overline{b},\overline{a})$ in the non-left-linear CSR with rules $a \to b$ and $f(\overline{x},\overline{x}) \to \ldots$, and $\ldots \hookleftarrow f(\overline{a}) \to f(\overline{b})$ in the non-canonical (and non-convective) CSR with rules $a \to b$ and $f(\overline{a}) \to c$.
- Generalising the first item, μ-steps can be *preponed*, i.e. $\twoheadrightarrow \cdot \hookrightarrow \subseteq \hookrightarrow \cdot \twoheadrightarrow$. This is a consequence of that $\mathrel{\to\!\!\!\!\!\!\shortmid\shortmid}_{\not\mu} \cdot \hookrightarrow \subseteq \hookrightarrow^{+} \cdot \mathrel{\to\!\!\!\!\!\!\shortmid\shortmid}_{\not\mu}$, which in turn may, using the methodology of [22], be seen to be a consequence of *orthogonality* between ${}_{\not\mu}\!\leftarrow$, (the *converse* of $\to_{\not\mu}$) and $\hookrightarrow$ and yielding $\mathrel{\to\!\!\!\!\!\!\shortmid\shortmid}_{\not\mu} \cdot \hookrightarrow \subseteq \hookrightarrow \cdot \mathrel{\to\!\!\!\!\!\!\shortmid\shortmid}$ from which we conclude by splitting and sequentialising the $\mathrel{\to\!\!\!\!\!\!\shortmid\shortmid}$-step into $\hookrightarrow$-steps (residuals of the frozen $\to_{\not\mu}$-step that have become active)[10] followed by the $\to_{\not\mu}$-steps (residuals that remained frozen).
  Another consequence (cf. [22]) is *$\mu,\not\mu$-factorisation* $\twoheadrightarrow \subseteq \mathrel{\hookrightarrow\!\!\!\!\twoheadrightarrow} \cdot \twoheadrightarrow_{\not\mu}$ [15, Theorem 5.7].
- Generalising the second item, μ-steps *commute* with non-μ-steps in the sense that $\hookleftarrow \cdot \twoheadrightarrow_{\not\mu} \subseteq \twoheadrightarrow \cdot \hookleftarrow$. This is a consequence of that $\hookleftarrow \cdot \mathrel{\to\!\!\!\!\!\!\shortmid\shortmid}_{\not\mu} \subseteq \mathrel{\to\!\!\!\!\!\!\shortmid\shortmid} \cdot \hookleftarrow$ which holds by orthogonality between μ-steps and non-μ-steps (using left-linearity and non-overlap) using standard residual theory, see [25, Chapter 8]. Note that as in the previous item, residual(s) of a non-μ-steps may become active.

(c) If $\mathcal{T}$ is a left-linear confluent TRS, μ a canonical replacement map, and $\hookrightarrow$ normalising, then $\hookrightarrow$ is confluent *up to* non-μ-convertibility. To see this, note that for any peak $t \mathrel{\twoheadleftarrow\!\!\!\!\hookleftarrow} s \mathrel{\hookrightarrow\!\!\!\!\twoheadrightarrow} u$[11] normalisation of $\hookrightarrow$ yields a peak $t' \mathrel{\twoheadleftarrow\!\!\!\!\hookleftarrow} t \mathrel{\twoheadleftarrow\!\!\!\!\hookleftarrow} s \mathrel{\hookrightarrow\!\!\!\!\twoheadrightarrow} u \mathrel{\hookrightarrow\!\!\!\!\twoheadrightarrow} u'$ for some

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[9] In [8] the connexion to the abstract approaches to normalisation in the literature was not made, this property was called *backward invariance of $\to_\mu$-normal forms* and established under the additional (superfluous) condition of level-decreasingness [8, Lemma 5].

[10] Qua abstract properties, being active is different from being needed [13] in that non-neededness *is* preserved by residuation, cf [25, Section 9.2]. In CSR non-0-collapsingness (see Lemma 21) is needed to guarantee that being frozen is preserved by residuation.

[11] Beware that we use $\mathrel{\hookrightarrow\!\!\!\!\twoheadrightarrow}$ to denote the reflexive–transitive closure of single-step context-sensitive rewriting $\hookrightarrow$. This differs from the meaning (layered rewriting) assigned to it in CSR [16]. We feel deviating from the latter is justified (despite the topic being CSR), since there is a long tradition in the rewriting literature [2, 25], to employ *double*-headed arrows to denote the reflexive–transitive closure of the relation / rewrite system denoted by the corresponding *single*-headed arrow. For instance, $\twoheadrightarrow_\beta$ is often used in the λ-calculus literature to denote many-step β-reduction. We follow this tradition here.

$\hookrightarrow$-normal forms t', s'. By confluence of $\hookrightarrow$, there is a valley $t' \hookrightarrow\!\!\!\!\twoheadrightarrow r \twoheadleftarrow\!\!\!\!\hookleftarrow u'$ for some term r. By the previous item all steps in the valley are non-μ-steps, from which we conclude.[12]

(d) Independently, Nao Hirokawa showed [9] Lemma 3 and also the Z-property for an outside–in defined bullet function, under (among others) the assumptions of canonicity and level-decreasingness originating from [8], which we have re(placed / laxed) in Lemma 3 to convectivity. We think the bullet functions coincide (extensionally) on finite terms for canonical replacement maps, but that they diverge for convective replacement maps or infinite terms.

(e) One of the novel results of [10] is that the full parallel-outermost strategy[13] $\twoheadrightarrow_{\mathsf{po}}$ is a hyper-normalising parallel strategy for $\rightarrow$, for $\mathcal{T}, \mu$ with $\mathcal{T}$ a left-linear confluent TRS, μ a canonical replacement map, and $\hookrightarrow$ terminating. As shown there, the result is a consequence of hyper-normalisation of layered CSR [16], allowing to perform $\hookrightarrow$-steps in in subterms if all layers on the path to the root are in $\hookrightarrow$-normal form.

Following up on Section 1 again, note that the proof strategy implicit in employing layered CSR is analogous to the explicit way of proving (hyper-)normalisation of the needed strategy from (hyper-)*head* normalisation [20, 25][14] and that since $\hookrightarrow$-normal forms in CSR are analogous to terms-in-head-normal-form in TRSs, layered CSR [16] could alternatively have been described as the *context free* $\hookrightarrow$-strategy [25, Definition 9.1.29].[15]

To flesh out this intuition, note first that under the assumptions the *relative* rewrite [6] system $\hookrightarrow/\rightarrow := \twoheadrightarrow \cdot \hookrightarrow \cdot \twoheadrightarrow$ is terminating as follows from $\hookrightarrow$-preponement (see item **(b)**) and termination of $\hookrightarrow$. From that, we immediately obtain hyper-*active*-normalisation, the property that always eventually contracting an active redex, i.e. performing a $\hookrightarrow$-step, yields a term in $\hookrightarrow$-normal form. This is the analogon of hyper-(head-)normalisation of the (head-)needed strategy.

By confluence of $\hookrightarrow$ up to non-μ-convertibility (see item **(c)**) we moreover obtain that the layers at depth 0 of all $\hookrightarrow$-normal forms are the same, with their respective arguments $\rightarrow$-convertible, hence $\rightarrow$-joinable, below the top layer. From this we conclude by an easy induction (since terms hence the normal form has a finite number of layers) that the context free $\hookrightarrow$-strategy is hyper-normalising as desired. This extends to a proof of infinitary hyper-normalisation by co-induction, but then *fairness*[16] (combined with the

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[12] In related work of ours (which we will pursue elsewhere) we extended the normalisation-by-random-descent results of [24] to a method for showing *head*-normalisation. The similarity with that work, cf. Section 1, is that in an infinitary / co-inductive setting one can in general not expect to have confluence, but it suffices to produce the same *head* / *top layer*, with, e.g., *convertibility* of corresponding arguments of the head / top layer guaranteeing that one can iterate the process on those arguments, yielding ω-confluence. Think of *different* algorithms for producing the decimal expansion of π (possibly with different *rates of convergence*). For instance, the TRS $\mathcal{T}$ with rules $\pi \to a$, $\pi \to s(a)$, $a \to s(s(a))$ is ω-confluent, but not confluent; see Example 28**(m)** and cf. the work of Blom, Ariola, and Klop.

[13] Huet introduced in [12] the notation $\dashrightarrow\!\!\!\!\!\shortparallel$ for *parallel* rewriting associated to a TRS $\mathcal{T}$, allowing to contract an *arbitrary* number of redexes at *parallel* positions. Unfortunately that same notation is sometimes used for (what we call) *full* parallel rewriting, allowing to contract only a *maximal* number of parallel redexes. We suggest to avoid conflating both no(ta)tions, and propose to employ the notation $\twoheadrightarrow$ for the latter instead, with the notation already suggesting that $\twoheadrightarrow$ is a *full* version of $\dashrightarrow\!\!\!\!\!\shortparallel$. This is analogous to that we use $\multimap\!\!\!\rightarrow$ to denote the full version (contracting a *maximal* number of non-overlapping redex-patterns) of *multi*step rewriting $\circ\!\!\!\rightarrow$ (contracting an *arbitrary* number of such) in our work [25, 23]. (Note that just as $\multimap\!\!\!\rightarrow$ is deterministic for TRSs without *critical pairs*, $\twoheadrightarrow$ is deterministic for system without *overlay critical pairs*.)

[14] But note that the outermost-fair strategy is normalising but need not be head-normalising for *weakly* orthogonal TRSs [21], cf. [25, Example 9.3.11].

[15] The name *context free* meshes well with $\hookrightarrow$ itself being *context sensitive* (but *not* a $\rightarrow$-strategy [25]).

[16] Without fairness, the context free $\hookrightarrow$-strategy would allow to rewrite a for the CSR with rule $a \to c(\overline{a}, \overline{a})$, into the tree $\ell := c(\ell, a)$ by always selecting the leftmost redex. Fairness overcomes this, yielding the

pigeon hole principle) is required as usual.[17]

(f) If $\mathcal{T}, \mu$ is level-decreasing and μ canonical, then the combination of items **(a)** and **(e)** yields that for a peak where the depth of its steps is bounded by n, successively applying $\bullet$ at depths $0, \ldots, n$ to its source yields a common reduct, giving a simple alternative proof of [8, Theorem 2].[18] This alternative proof, though simple, still uses [8, Lemma 1] to obtain confluence of $\hookrightarrow$, which is an *assumption* (assumption **(i)**) here but a *consequence* of further assumptions put forward in [8], cf. also [17].

(g) For the *non-left-linear* CSR with rules $a \to b$ and $f(\overline{x}, \overline{x}) \to c$, the lemma fails on $f(\overline{a}, \overline{a})$.

Assumption **(ii)** ensures $\hookrightarrow$ has the Z-property for *bullet* map $\bullet$ by [23, Lemma 11]. That bullet map is *extensive* for $\hookrightarrow$, i.e. $t \hookrightarrow\!\!\!\!\rightarrow t^{\bullet}$ [23, Definition 4]. We show $\to$ has the Z-property under assumptions **(i)** and **(ii)** for some bullet map $\circledbullet$ based on $\bullet$. To define $\circledbullet$ we use that any term can be uniquely decomposed into its *active* layer at depth 0 w.r.t. μ,[19] and its *frozen* arguments at depth 1. Accordingly, we write $C\langle\vec{t}\rangle$ to denote such a unique decomposition, where C is the active layer and $\vec{t}$ the (vector of) frozen arguments.

▶ **Definition 5.** *The* layering $\circledbullet$ *(of $\bullet$) is inductively defined by* $C\langle\vec{t}\rangle^{\circledbullet} := C\langle\vec{t}^{\circledbullet}\rangle^{\bullet}$.

▶ Remark 6.
- The idea of the layering $\circledbullet$ is the same as that of (super)development bullet functions[20] $\bullet$ in [23], namely to first recursively apply the map to the *sub*-layers, and then perform an appropriate action on the *top*-layer. As observed in Remark 2**(c)**, for orthogonal TRSs the convective replacement map μ^{con} is empty, so all arguments are frozen. If moreover no rhs is unifiable with any lhs (entailing the TRS is non-collapsing), so that contracting a redex cannot *create* another $\hookrightarrow$-step, then the layering $\circledbullet$ even *coincides* with the superdevelopment bullet function of [23].
- It would be interesting to formulate and prove a *preservation* result, more precisely to show that, under suitable conditions, a bullet map $\bullet$ having the Z-property for the rewrite system $\hookrightarrow$ on single layers, induces its layering $\circledbullet$ also has the Z-property for $\to$. The proof method below is not suitable for that, since it hinges on that $\bullet$ be *idempotent*, $(t^{\bullet})^{\bullet} = t^{\bullet}$, a property which almost forces that $\bullet$ maps terms to their $\hookrightarrow$-normal form, which is much stronger than just having the Z-property (normal forms as obtained by $\bullet$ are *maximal* upper bounds, whereas for the Z-property typically non-maximal upper bounds are used [23]; e.g. also $\circledbullet$ typically will neither be idempotent nor maximal).

Our first result bears witness to the inside–out, layer-wise nature of the layering $\circledbullet$ of $\bullet$.

▶ **Lemma 7.** $C[\vec{t}^{\circledbullet}] \twoheadrightarrow C[\vec{t}]^{\circledbullet}$

Proof. By induction and cases on C. The base cases $C = \Box$ and $C = x$ being trivial, suppose C has shape $f(\vec{C})$ and decompose $\vec{t}$ accordingly. We conclude to $C[\vec{t}^{\circledbullet}] = f(\overrightarrow{C[\vec{t}^{\circledbullet}]}) \twoheadrightarrow f(\overrightarrow{C[\vec{t}]^{\circledbullet}}) \twoheadrightarrow f(\overrightarrow{C[\vec{t}]})^{\circledbullet} = C[\vec{t}]^{\circledbullet}$ by, respectively, the decomposition of $C[\vec{t}]$, the induction hypothesis for $\vec{C}$ and closure under contexts of $\to$, the claim that $g(\vec{s}^{\circledbullet}) \twoheadrightarrow g(\vec{s})^{\circledbullet}$ for all g and $\vec{s}$, and by definition of the decomposition again.

To prove the claim, first observe that $g(\vec{s}^{\circledbullet}) \twoheadrightarrow g(\vec{s}^{\circledbullet})^{\bullet}$ by extensivity of $\bullet$ and $\hookrightarrow \subseteq \to$. Therefore, to conclude it suffices to show $g(\vec{s}^{\circledbullet})^{\bullet} = g(\vec{s})^{\circledbullet}$. To that end, let $g(\vec{s})$ uniquely

infinite normal form $t := c(t, a)$ as desired.

[17] Alternatively, *transfinite* reductions could be employed to go beyond ω-length reduction.

[18] Below we show the requirements of level-decreasingness and μ being canonical to be too restrictive.

[19] In [16] this is called the *maximal replacing layer* and denoted by MRC^{μ}.

[20] Developments and superdevelopments are also known as *full* multisteps and supersteps.

decompose as $g(\overrightarrow{D[\vec{u}]})$ with for $i \in \mu(g)$, $D_i\langle\vec{u_i}\rangle$ the unique decomposition of s_i, and for $i \notin \mu(g)$, $D_i = \Box$ and $\vec{u_i} = s_i$. Hence $g(\vec{s})^{\circledbullet} = g(\overrightarrow{D[\vec{u}^{\circledbullet}]})^{\bullet}$ per construction of the decomposition and by definition of $\circledbullet$. To conclude to $g(\vec{s}^{\circledbullet})^{\bullet} = g(\vec{s})^{\circledbullet} = g(\overrightarrow{D[\vec{u}^{\circledbullet}]})^{\bullet}$ it then suffices to show that $g(\vec{s}^{\circledbullet})$ and $g(\overrightarrow{D[\vec{u}^{\circledbullet}]})$ are $\hookrightarrow$-convertible since $\hookrightarrow$ is complete by assumption **(ii)**. Convertibility follows from that for each active argument $i \in \mu(g)$ we have that s_i uniquely decomposes as $D_i\langle\vec{u_i}\rangle$ so that $s_i^{\circledbullet} = D_i\langle\vec{u_i}^{\circledbullet}\rangle^{\bullet}$ hence $s_i^{\circledbullet}$ and $D_i\langle\vec{u_i}^{\circledbullet}\rangle$ are $\hookrightarrow$-convertible and by i being active this extends to the respective ith arguments of $g(\vec{s}^{\circledbullet})$ and $g(\overrightarrow{D[\vec{u}^{\circledbullet}]})$, and from that for each frozen argument $i \notin \mu(g)$ we have by definition of D_i and $\vec{u_i}$ that $s_i^{\circledbullet} = D_i[\vec{u_i}^{\circledbullet}]$. ◀

Note $\circledbullet$ is extensive for $\rightarrow$, as an instance / consequence of Lemma 7 (for $\vec{t}$ empty).

▶ **Theorem 8.** *$\rightarrow$ has the Z-property for $\circledbullet$.*

Proof. We have to show that if $\phi : t \rightarrow s$ is a TRS step, then there are reductions $s \twoheadrightarrow t^{\circledbullet}$ and $t^{\circledbullet} \twoheadrightarrow s^{\circledbullet}$, giving rise to the Z in [23, Figures 1 and 5]. This we prove by induction on the decomposition $C\langle\vec{t}\rangle$ of the source t of ϕ and by cases on whether or not ϕ is a μ-step.

- if $t \hookrightarrow s$, then by definition of $\circledbullet$ and extensivity of $\circledbullet$, there is a reduction $t \twoheadrightarrow t^{\circledbullet}$ that decomposes into a reduction $\gamma : C\langle\vec{t}\rangle \twoheadrightarrow C\langle\vec{t}^{\circledbullet}\rangle$ with steps at depth at least 1, followed by a reduction $\delta : C\langle\vec{t}^{\circledbullet}\rangle \hookrightarrow\!\!\!\!\rightarrow C\langle\vec{t}^{\circledbullet}\rangle^{\bullet} = t^{\circledbullet}$ with steps at depth 0. Since ϕ is a step at depth 0, assumption **(i)** yields it and its residuals (after any prefix of γ) are orthogonal to (the corresponding suffix of) γ, giving rise by standard residual theory [25, Chapter 8] to a valley completing the peak between ϕ and γ that comprises a step $\phi/\gamma : C\langle\vec{t}^{\circledbullet}\rangle \hookrightarrow u$ and reduction $\gamma/\phi : s \twoheadrightarrow u$ for some term u.
  To conclude to $s \twoheadrightarrow t^{\circledbullet}$ we compose $\gamma/\phi : s \twoheadrightarrow u$ with the $\hookrightarrow$-reduction (lifted to a $\rightarrow$-reduction using $\hookrightarrow \subseteq \rightarrow$) of its target u to $\hookrightarrow$-normal form, which is $t^{\circledbullet}$ since $t^{\circledbullet} = C\langle\vec{t}^{\circledbullet}\rangle^{\bullet} = u^{\bullet}$ by definition respectively ϕ/γ and completeness of $\hookrightarrow$.
  To conclude to $t^{\circledbullet} \twoheadrightarrow s^{\circledbullet}$, we claim that u has shape $E[\vec{u}^{\circledbullet}]$ and s has shape $E[\vec{u}]$ for some context E and vector of terms $\vec{u}$. Then, composing $\phi/\gamma : C\langle\vec{t}^{\circledbullet}\rangle \hookrightarrow u$ with $u = E[\vec{u}^{\circledbullet}] \twoheadrightarrow E[\vec{u}]^{\circledbullet} = s^{\circledbullet}$ obtained by Lemma 7, yields $C\langle\vec{t}^{\circledbullet}\rangle \twoheadrightarrow s^{\circledbullet}$. From this we conclude to $t^{\circledbullet} = C\langle\vec{t}^{\circledbullet}\rangle^{\bullet} \twoheadrightarrow (s^{\circledbullet})^{\bullet} = s^{\circledbullet}$ by Lemma 3 and idempotence of $\bullet$.
  It remains to prove the claim that u has shape $E[\vec{u}^{\circledbullet}]$ and s has shape $E[\vec{u}]$ for some context E and vector of terms $\vec{u}$. The idea is that both C and ℓ are preserved under non-μ-steps, so their *join* is so too, and we set E be the result of contracting ℓ in the join. Formally, we construct E as follows. Let $\varsigma :=$ **let** $X = C[\vec{x}]$ **in** $X(\vec{t})$ be the *cluster* [11] corresponding to the occurrence of the context C in t, and let ζ be the cluster of shape **let** $Y = \ell$ **in** $\ldots$ corresponding to the occurrence in t of the left-hand side ℓ of the rule $\ell \rightarrow r$ contracted in the step $\phi : t \hookrightarrow s$. Their join $\xi := \varsigma \sqcup \zeta$ has shape **let** $Z = D[\vec{z}]$ **in** $Z(\vec{u})$ for some context D and terms $\vec{u}$, by ς being a root cluster of ς having overlap with ζ.
  Per construction of ξ and by the TRS $\mathcal{T}$ being left-linear, there is some step ψ from $D[\vec{z}]$ contracting the occurrence of ℓ, such that ϕ is a substitution instance of ψ.[21] Then we define E from the target of ψ writing that uniquely as $E[\vec{w}]$ for $\vec{w}$ comprising the replicated variables of $\vec{z}$, so that $\psi : D[\vec{z}] \hookrightarrow E[\vec{w}]$. In turn, we define $\vec{u}$ from the target

[21] Using traditional unification-speak D can be described as being obtained by unifying the occurrence of the left-hand side ℓ with the context C (both linear and renamed apart). E is then the result of contracting the ℓ-redex in D. We prefer to employ the lattice-theoretic language developed in [11] as that is based on encompassment which encompasses both the subsumption (prefix; unification) and the superterm (suffix) orders employed in such traditional accounts, and moreover avoids context-talk which is imprecise here since D and E are not simply contexts, but linear terms; in particular, the names of the holes in E do matter.

s of $\phi : t \hookrightarrow s$, noting the latter can be written as the unique substitution instance $E[\vec{w}]^\upsilon = E[\vec{u}]$ of the target $E[\vec{w}]$ of ψ, for substitution υ mapping z_i to u_i such that $\phi = \psi^\upsilon$. Per construction, $t = D[\vec{z}]^\upsilon$ and $s = E[\vec{w}]^\upsilon = E[\vec{u}]$.
Finally, we must show that $u = E[\vec{u}^{\circledast}]$. To that end, note that any $\hookrightarrow$-step ϕ' of shape ψ^σ for term substitution σ, is orthogonal to any non-μ-step χ having the same source, as (the redex-pattern of) χ can neither have overlap with ς by χ being non-μ, nor have overlap with ζ by assumption **(i)** using that ψ is at depth 0 and χ at depth at least 1, so χ cannot have overlap with their join $\varsigma \sqcup \zeta$ either. Thus, χ is of shape $D[\vec{z}]^\tau$ for some step-substitution[22] τ, and $\chi/\phi' = E[\vec{w}]^\tau$ and $\phi'/\chi = \psi^{\tau'}$ with τ' the step-substitution such that $\tau'(z_i)$ is the target of $\tau(z_i)$, for all i.
By induction on the length of γ, we obtain from the above that the reduction $\gamma : t = C\langle\vec{t}\rangle \twoheadrightarrow C\langle\vec{t}^{\circledast}\rangle$, comprises only steps that are substitution instances of $D[\vec{z}]$ so that $C\langle\vec{t}^{\circledast}\rangle$ is as well. In particular note that each reduction from t_i to $t_i^\bullet$ does not change its top part (if any) overlapping the occurrence of ℓ, so is the same as that top part where all its arguments have been reduced to $\circledast$-normal form. That is, $C\langle\vec{t}^{\circledast}\rangle$ has shape $D[\vec{z}]^{\upsilon^{\circledast}}$. By the above, u then has shape $E[\vec{w}]^{\upsilon^{\circledast}} = E[\vec{u}^{\circledast}]$ as common target of ϕ/γ and γ/ϕ, as claimed.

- if $t \to s$ is not a μ-step then $s = C\langle\vec{s}\rangle$ with $t_i \to s_i$ for some i and $t_j = s_j$ for all $j \neq i$. Then the Z-property holds for $\vec{s}$, i.e. $\vec{s} \twoheadrightarrow \vec{t}^{\circledast} \twoheadrightarrow \vec{s}^{\circledast}$ since by the IH $s_i \twoheadrightarrow t_i^{\circledast} \twoheadrightarrow s_i^{\circledast}$, and $s_j \twoheadrightarrow t_j^{\circledast} = s_j^{\circledast}$ for all $j \neq i$ by extensivity of $\circledast$. We conclude to $s = C\langle\vec{s}\rangle \twoheadrightarrow C\langle\vec{t}^{\circledast}\rangle \twoheadrightarrow C\langle\vec{t}^{\circledast}\rangle^\bullet = t^{\circledast} \twoheadrightarrow C\langle\vec{s}^{\circledast}\rangle^\bullet = s^{\circledast}$, using that the Z-property holds for $\vec{s}$ by the IH and closure of $\to$ under contexts for the first reduction, extensivity of $\bullet$ and $\hookrightarrow \subseteq \to$ for the second, and Z for $\vec{s}$ and closure under contexts and preservation of $\twoheadrightarrow$ by $\bullet$ for the third. ◀

3 ω-confluence without confluence

In this section we are interested in transferring local confluence of $\to$ to ω-confluence of $\to$. To that end we assume throughout that $\mathcal{T}, \mu$ is a CSR with $\mathcal{T}$ a left-linear locally confluent TRS, μ is canonical, and $\hookrightarrow$ is terminating, unless stated otherwise. Although these conditions do not entail confluence of $\hookrightarrow$ as shown in [8, Example 3] (see below), we show that they do entail ω-confluence. We first show that under the above assumptions any finite term t has a *potentially* infinite normal form, and that the latter is reachable from any reduct of t. Next, we extend our results to *actually* infinite reductions on infinite terms, in particular we show that ω-reduction has the triangle property for the context sensitive bullet-function $\circ$, mapping t to its (possibly infinite) normal form.

▶ **Definition 9.** *Let $\hookrightarrow\!\!\!\!\!\!\Rightarrow$ allow to contract a $\hookrightarrow$ step only in a layer of minimal depth.*

Then $\hookrightarrow\!\!\!\!\!\!\Rightarrow$ is a $\to$-strategy (layered CSR; see Remark 4**(e)**) since for a term not in $\to$-normal form, there is *some* minimal layer not in normal form. Observe that if a step at depth d occurs in a $\hookrightarrow\!\!\!\!\!\!\Rightarrow$-reduction, then all steps occurring later in it have depths $\geq d$ due to the canonicity and left-linearity assumptions. To set the stage, we first show that $\hookrightarrow\!\!\!\!\!\!\Rightarrow$ always produces a normal form and give an *existential* version of the *universal* Lemma 3.[23]

▶ Remark 10. This holds irrespective of the rules being collapsing or not, a condition often found in the study of infinitary rewriting, cf. [25, Chapter 12]. The reason for this is that

[22] A substitution τ such that for all i, $\tau(z_i)$ either is a single step or a term.
[23] The switch from universal to existential is needed since left-linearity and canonicity do not (yet) suffice for uniqueness of $\hookrightarrow$-normal forms; the map $\bullet$ in Lemma 3 is not well-defined without more.

the assumption that $\hookrightarrow$ be terminating already precludes collapsing the layer at depth 0 arbitrarily often, on *finite* terms that is; cf. Remark 17.

▶ **Lemma 11.** *For any maximal $\hookrightarrow\!\!\!\!\Rightarrow$-reduction and any depth d, there is a tail of the reduction in which the first d layers are in $\hookrightarrow$-normal form.*

Proof. Consider a maximal $\hookrightarrow\!\!\!\!\Rightarrow$-reduction from t. By induction on d, with the base case being trivial because we may take the reduction itself as witness. In the step case, the induction hypothesis yields a tail, say from s, of the reduction in which the first d layers are in $\hookrightarrow$-normal form. By the observation the latter property is preserved by later steps, hence it suffices to show that there is a tail of *that* tail, in which all the layers at depth d are in normal form. If this were not the case, there would by maximality be infinitely many steps at depth d starting from s, entailing by s being finite and the pigeon hole principle that there would be infinitely many $\rightarrow$-steps at depth d in some layer of s. But that would entail that the corresponding subterm allowed infinitely many $\hookrightarrow$ steps, contradicting the assumption that $\hookrightarrow$ is terminating. ◀

▶ **Proposition 12.** *For all $\rightarrow$-reductions $t \twoheadrightarrow s$, there are terms t', s' in $\hookrightarrow$-normal form such that $t \hookrightarrow\!\!\!\!\twoheadrightarrow t'$, $s \hookrightarrow\!\!\!\!\twoheadrightarrow s'$ and $t' \twoheadrightarrow s'$ with all steps in the last at depth ≥ 1.*

Proof. By well-founded induction on t ordered by $\hookleftarrow$.

If t is in $\hookrightarrow$-normal form, we may trivially set $t' := t$ and $s' := s$. Otherwise we distinguish cases on whether or not $t \twoheadrightarrow s$ contains a step at depth 0.

If it does, then by Remark 4**(b)** the μ-step can be preponed before the non-μ-steps before it (only using left-linearity and canonicity) in $t \twoheadrightarrow s$, yielding $t \hookrightarrow u$ and $u \twoheadrightarrow s$ for some term u. we conclude by the IH for $u \twoheadrightarrow s$.

If it doesn't, choose any $\hookrightarrow$-step $t \hookrightarrow t'$ from t. Per assumption, that step (at depth 0) is orthogonal to the $\rightarrow$-reduction (at depth ≥ 1) $t \twoheadrightarrow s$. Orthogonally projecting them over each other (possible by left-linearity and canonicity as the latter entails convectivity) yields $s \hookrightarrow s'$ and $t' \twoheadrightarrow\!\!\!\!\rightarrow s'$ (by sequentialising for each step of $t \twoheadrightarrow s$ the parallel step that is its residual after $t \hookrightarrow t'$). We conclude by the IH for $t' \twoheadrightarrow s'$.[24] ◀

We did not yet employ local confluence of the TRS, i.e. of $\rightarrow$. It guarantees uniqueness of normal layers.

▶ **Lemma 13.** *If $t \twoheadrightarrow s, u$ with s, u in $\hookrightarrow$-normal form, then $s = C\langle\vec{s}\rangle$ and $u = C\langle\vec{u}\rangle$ for some layer C and $\rightarrow$-convertible $\vec{s}$, $\vec{u}$.*

Proof. By well-founded induction on t ordered by $\hookleftarrow$. Suppose $t \twoheadrightarrow s, u$ with s, u in $\hookrightarrow$-normal form. By postponement of non μ-steps after μ-steps (cf. the proof of Proposition 12), the reductions in the peak *factorise* as $t \hookrightarrow\!\!\!\!\twoheadrightarrow \hat{s} \twoheadrightarrow_{\not\mu} s$ respectively $t \hookrightarrow\!\!\!\!\twoheadrightarrow \hat{u} \twoheadrightarrow_{\not\mu} u$ for some terms $\hat{s}$ and $\hat{u}$. Since $\rightarrow_{\not\mu}$-steps cannot create μ-normal forms (Remark 4**(b)**), we have both $D\langle\vec{\hat{s}}\rangle = \hat{s}$ and $s = D\langle\vec{s}\rangle$ for some D and $\rightarrow$-convertible $\vec{\hat{s}}$, $\vec{s}$ (in fact the former $\rightarrow$-reduce to the latter), and $E\langle\vec{\hat{u}}\rangle = \hat{u}$ and $u = E\langle\vec{u}\rangle$ for some E and $\rightarrow$-convertible $\vec{\hat{u}}$, $\vec{u}$. Hence it suffices to show $D = E$ and $\vec{\hat{s}}$ and $\vec{\hat{u}}$ are $\rightarrow$-convertible.

To see this holds we distinguish cases on whether or not t is in $\hookrightarrow$-normal form. If it is, then we conclude since then we must have $\hat{s} = t = \hat{u}$. Otherwise, the $\hookrightarrow$-reductions in the

[24] Although in this case all steps in $t \twoheadrightarrow s$ are at depths ≥ 1 per assumption, this need hold true for $t' \twoheadrightarrow s'$, as we did not assume any restrictions on levels in rules here.

peak must both be non-empty since $\hat{s}, \hat{u}$ *are* in $\hookrightarrow$-normal form, so the $\hookrightarrow$-reductions can be written as $t \hookrightarrow s' \hookrightarrow\!\!\!\!\rightarrow \hat{s}$ respectively $t \hookrightarrow u' \hookrightarrow\!\!\!\!\rightarrow \hat{u}$, for some terms s', u'.

By the local confluence assumption for $\to$ (and $\hookrightarrow \subseteq \to$) for the peak $s' \hookleftarrow t \hookrightarrow u'$, there is a $\to$-valley $s' \twoheadrightarrow r \twoheadleftarrow u'$ for some term r, which we, by the assumption that $\hookrightarrow$ is terminating, may assume to be in $\hookrightarrow$-normal form, say it decomposes as $C\langle\vec{r}\rangle$. By the IH for the peak $s' \twoheadrightarrow \hat{s}, r$ we have that $D = C$ and that $\vec{\hat{s}}$ and $\vec{r}$ are $\to$-convertible, and by the IH for the peak $u' \twoheadrightarrow r, \hat{u}$ we have that $C = E$ and that $\vec{r}$ and $\vec{\hat{u}}$ are $\to$-convertible, so we conclude by transitivity to $D = E$ and to convertibility of $\vec{\hat{s}}, \vec{\hat{u}}$, respectively. ◀

▶ Remark 14. For the proof technique applied in the proof of Lemma 13, cf. Remark 4**(c)**.

▶ **Theorem 15.** *From all convertible terms there are reductions that for all depths d have tails having their first d layers in normal form and common.*

Proof. We claim that if terms t, s are $\to$-convertible, they can be $\to$-reduced to $C\langle\vec{t}\rangle$ and $C\langle\vec{s}\rangle$ respectively, for some C in normal form and $\to$-convertible $\vec{t}, \vec{s}$. From the claim we conclude, since then repeating the procedure on t_i, s_i for each i, produces a sequence of reducts of t, s that have an ever increasing number of layers in normal form in common.

We prove the claim by induction on the number of peaks in a conversion between t, s.

If there is no peak, then the conversion is a valley $t \twoheadrightarrow u \twoheadleftarrow s$ for some term u, and we trivially conclude by $\hookrightarrow$-reducing u to $\hookrightarrow$-normal form $C\langle\vec{u}\rangle$, and setting $\vec{t} := \vec{s} := \vec{u}$.

Suppose there is peak, say $t \twoheadleftarrow u \twoheadrightarrow t'$ with t' convertible to s with fewer peaks. Then by the IH t', s can be $\to$-reduced to $C\langle\vec{t'}\rangle$ and $C\langle\vec{s}\rangle$ respectively, for some C and convertible $\vec{t'}, \vec{s}$. Since by the termination assumption t reduces to some $\hookrightarrow$-normal form $\hat{t}$, Lemma 13 applied to $u \twoheadrightarrow \hat{t}, C\langle\vec{t'}\rangle$ yields that $\hat{t}$ has shape $C\langle\vec{t}\rangle$ for (the same C and) terms $\vec{t}$ that are convertible to $\vec{t'}$. We conclude since $t \twoheadrightarrow \hat{t} = C\langle\vec{t}\rangle$ and $s \twoheadrightarrow C\langle\vec{s}\rangle$, and $\vec{t}, \vec{s}$ are convertible by transitivity of convertibility. ◀

Combining the theorem with Lemma 11 yields that $\hookrightarrow\!\!\!\!\rightarrow$ is an *infinitary cofinal* strategy. This is not to be interpreted in the standard sense that any $\to$-reduct s of a term t reduces further to some term in any $\hookrightarrow\!\!\!\!\rightarrow$-reduction from t (that may not hold as our examples below show). We do have though a reduction from s whose terms share an ever increasing number of layers in normal form with the $\hookrightarrow\!\!\!\!\rightarrow$-reduction from t.

If the formulations in the above are a bit awkward, this is due to that we have thus far only employed the *potential* not the *actual* infinite. We now deal with the latter, by extending the above reasoning. We adopt infinitary rewriting for iTRSs as in [25, Chapter 12]. More precisely, while allowing to rewrite infinite terms we still assume left- and right-hand sides of rules to be finite, which allows us to restrict attention to strongly converging reductions of length at most ω, denoted by triply-replicating arrow-heads like $\twoheadrightarrow\!\!\!\!\rightarrow$, since by [25, Theorem 12.7.1] reductions of greater ordinal length can be *compressed* to such.

▶ **Definition 16.** *An iTRS is ω-*confluent *[3] if $\twoheadrightarrow\!\!\!\!\rightarrow$ has the diamond property from finite terms, and has the ω-*angle *property if $\twoheadrightarrow\!\!\!\!\rightarrow$ has the angle property [23] from finite terms.*

As in the finite case [25, Chapter 1], the ω-angle property entails ω-confluence.

▶ Remark 17. It is perfectly reasonable [25, Chapter 12] to try to lift the restriction that the sources of diamonds and angles be finite terms. Care is required though since such choices typically do affect properties. For instance, for the collapsing rule $c(\overline{x}) \to x$, $\hookrightarrow$ is terminating on finite terms (our assumption here throughout), but not on the infinite term $t := c(\overline{t})$. Cf. also the classical example [25, Chapter 12] of the iTRS with rules $a(x) \to x$ and $b(x) \to x$ which is complete on finite terms, hence ω-confluent, but non-infinitary-terminating and non-infinitary-confluent since the peak $t \twoheadrightarrow\!\!\!\!\rightarrow s, u$ for the infinite terms $t := a(b(t))$, $s := a(s)$ and $u := b(u)$ is not infinitary joinable. Similarly, but stronger, the infinite term $t_0 := p(s(s(p(p(p(\ldots))))))$ (formally defined by $t_n := s^n(u_{n+1})$ and $u_n := p^n(t_{n+1})$) in the *weakly* orthogonal iTRS due to Klop [4] with rules $p(s(x)) \to x$ and $s(p(x)) \to x$, infinitary reduces to *distinct* infinite normal forms $s^\omega := s(s^\omega)$ and $p^\omega := p(p^\omega)$;[25] see Example 28(**n**).

▶ **Theorem 18.** *The ω-angle property holds.*

Proof. We show that $\twoheadrightarrow\!\!\!\!\rightarrow$ has the triangle property for the map $\circ$ that maps any finite term t to its, possibly infinite, $\hookrightarrow$-normal form denoted by t°, existing uniquely by the above.

To show this, consider an ω-reduction $t \twoheadrightarrow\!\!\!\!\rightarrow s$. We claim then $t \hookrightarrow\!\!\!\!\rightarrow t'$, $s \hookrightarrow\!\!\!\!\rightarrow s'$ for some t', s' in $\hookrightarrow$-normal form such that $t' \twoheadrightarrow\!\!\!\!\rightarrow s'$ with all steps at depth ≥ 1. This suffices since by the above then t° and t' have the same layer at depth 0, which per construction is the same as that of s', and for the arguments the assumption holds again. Hence repeating the construction yields $\hookrightarrow$-reductions through $t, t', t'', \ldots$ and $s, s', s'', \ldots$ respectively, whose terms share $0, 1, 2, \ldots$ layers in normal form, hence also with t°. That is, both are strongly converging ω-reductions to t°.

To prove the claim we proceed as in the proof of Proposition 12, by well-founded induction on t ordered by $\hookleftarrow$.

If t is in $\hookrightarrow$-normal form, we may trivially set $t' := t$ and $s' := s$. Otherwise we distinguish cases on whether or not $t \twoheadrightarrow\!\!\!\!\rightarrow s$ contains a step at depth 0.

If it does, then by $t \twoheadrightarrow\!\!\!\!\rightarrow s$ being of length at most ω the first $\hookrightarrow$-step takes place at a finite index in the ω-reduction. Per our assumptions all terms in it up to that index are finite, so factorisation applied to it yields $t \hookrightarrow u$ and $u \twoheadrightarrow\!\!\!\!\rightarrow s$ for some (finite) term u. we conclude by the IH for $u \twoheadrightarrow\!\!\!\!\rightarrow s$.

If it doesn't, choose any $\hookrightarrow$-step $t \hookrightarrow t'$ from t. Per assumption, that step (at depth 0) is orthogonal to the ω-reduction (at depth ≥ 1) $t \twoheadrightarrow\!\!\!\!\rightarrow s$. Orthogonally projecting them over each other (possible by left-linearity and canonicity as the latter entails convectivity) yields $s \hookrightarrow s'$ and $t' \twoheadrightarrow\!\!\!\!\rightarrow s'$ (by sequentialising for each step of $t \twoheadrightarrow\!\!\!\!\rightarrow s$ the parallel step that is its residual after $t \hookrightarrow t'$). We conclude by the IH for $t' \twoheadrightarrow\!\!\!\!\rightarrow s'$.[26] ◀

In the above, the assumption that the TRS be locally confluent was only used in the proof of Lemma 13. It may be relaxed, while preserving the conclusion of the lemma.

▶ **Definition 19.** *A CSR is* 0-locally confluent *if for for every local peak $s \hookleftarrow t \hookrightarrow u$ there is a C in $\hookrightarrow$-normal form such that $s \twoheadrightarrow C\langle\vec{s}\rangle$ and $u \twoheadrightarrow C\langle\vec{u}\rangle$ with $\to$-convertible $\vec{s}, \vec{u}$.*

Note that for left-linear CSRs with canonical replacement map, local confluence entails 0-local confluence as in the proof of the lemma by $\hookrightarrow$-normalising the common reduct, that that proof factors through 0-local confluence, yielding:

▶ **Corollary 20** (to the proof of Theorem 18)**.** *The ω-angle property holds for any left-linear* 0*-locally confluent CSR having a canonical replacement map and terminating $\hookrightarrow$.*

Note that 0-local confluence is not decidable, because already local confluence is not (since $\to$ need not be terminating; termination of $\hookrightarrow$ does not bring much qua decidability as required

[25] This is a variation on the Grandi's series. This variation is nice in that it suffices to repeatedly cancel *adjacent* $+1$ and -1 to obtain distinct results; i.e. that is a matter of bracketing only.

[26] Although in this case all steps in $t \twoheadrightarrow\!\!\!\!\rightarrow s$ are at depths ≥ 1 per assumption, this need not hold true for $t' \twoheadrightarrow\!\!\!\!\rightarrow s'$, as we do not assume any restrictions on levels in rules here.

reductions can be easily 'hidden' inside frozen arguments). We identify a simple case, in the spirit of Huet's critical peak lemma, in which 0-local confluence follows from the same restricted to *critical* $\hookrightarrow$-peaks.

▶ **Lemma 21.** *If all rules are* non-0-collapsing*, that is, if for any rule and each variable that has level* ≥ 1 *in the lhs only occurs at levels* ≥ 1 *in the rhs, then* 0*-local confluence follows from the same for critical* $\hookrightarrow$*-peaks*

Proof. Consider a local peak $s \hookleftarrow t \hookrightarrow u$ and distinguish cases on whether or not the steps are orthogonal to each other. If they are, they commute and we conclude. Otherwise, the peak is obtained by closing some critical peak $\hat{s} \hookleftarrow \hat{t} \hookrightarrow \hat{u}$ under a substitution υ an an active context D. By assumption, there are a C in $\hookrightarrow$-normal form and $\rightarrow$-convertible $\vec{\hat{s}}, \vec{\hat{u}}$ such that $\hat{s} \twoheadrightarrow C\langle\vec{\hat{s}}\rangle$ and $\hat{u} \twoheadrightarrow C\langle\vec{\hat{u}}\rangle$. By $\rightarrow$ being closed under substitutions and contexts we obtain $s = D[\hat{s}^\upsilon] \twoheadrightarrow D[C^\upsilon\langle\vec{\hat{s}}^\upsilon\rangle]$ and $u = D[\hat{u}^\upsilon] \twoheadrightarrow D[C^\upsilon\langle\vec{\hat{u}}^\upsilon\rangle]$ with $\rightarrow$-convertible $\vec{\hat{s}}^\upsilon, \vec{\hat{u}}^\upsilon$. Let γ be the $\hookrightarrow$-reduction from $D[C^\upsilon\langle\vec{x}\rangle]$ to $\hookrightarrow$-normal form $E[\vec{y}]$ with $\vec{y}$ a permutation / replication of $\vec{x}$, which exists by the assumption that $\hookrightarrow$ is terminating. By non-0-collapsingness all $\vec{y}$ occur in E at levels ≥ 1. Let τ and σ map $\vec{x}$ to $\vec{\hat{s}}^\upsilon$ respectively $\vec{\hat{u}}^\upsilon$ so that $D[C^\upsilon\langle\vec{\hat{s}}^\upsilon\rangle] = D[C^\upsilon\langle\vec{x}^\tau\rangle]$ and $D[C^\upsilon\langle\vec{\hat{u}}^\upsilon\rangle] = D[C^\upsilon\langle\vec{x}^\sigma\rangle]$. Then projecting the conversion between $D[C^\upsilon\langle\vec{\hat{s}}^\upsilon\rangle]$ and $D[C^\upsilon\langle\vec{\hat{u}}^\upsilon\rangle]$ at levels ≥ 1 over γ yields a conversion between $E[\vec{y}^\tau]$ and $E[\vec{y}^\sigma]$ with steps 'within' the terms substituted for the $\vec{y}$, hence with steps at levels ≥ 1 again. ◀

▶ Remark 22. The above can be thought of as employing $\rightarrow$-convertibility as bisimulation.

4 Related work

As already observed above in Remark 4**(f)** our approach to confluence via the Z-property in Section 2 has (local) confluence of context-sensitive rewriting $\hookrightarrow$ as an *assumption*, whereas in e.g. [8, 17] that is a *consequence* of further ('more local') assumptions, in particular of level-decreasingness of $\mathcal{T},\mu$ and canonicity of μ in [8]. However, *any* way to establish local confluence of $\hookrightarrow$ suffices to apply our results in Section 2. For instance, as we will show now, it suffices to assume level-decreasingness only for variables *active* in the left-hand side, what we call 0-preservingness.

▶ Remark 23.
- 0-preservingness is obtained by specialising the *LHRV-condition* known from the literature, cf. [17, Definition 11], to the left-linear TRSs dealt with here [17, Proposition 13(3)]. We employ our naming as it already suggests that the condition is the weakening of level-decreasingness only restricting active variables.
- Despite that Lemma 24 below is a special case of [17, Theorem 30], arising by (additionally) assuming left-linearity and the absence of extended critical pairs [17, Definition 29], we present it as this specialisation is easy to state, understand and prove.[27]
- Local confluence of context-sensitive rewriting $\hookrightarrow$ may be established by *any* of the other techniques developed in [17], e.g. the one based on non-trivial instances of [17, Theorem 30] using extended critical pairs and automated reasoning to establish their $\hookrightarrow$-joinability.

(iii) $\mathcal{T},\mu$ is 0-*preserving* if, whenever a variable occurs at depth 0 in the left-hand side of a rule, then all its occurrences in the right-hand side are at depth 0 as well.

[27] The proof is obtained from that of [17, Theorem 30] by simply dropping the complex cases.

▶ **Lemma 24.** *If $\mathcal{T},\mu$ is a left-linear CSR satisfying assumptions **(i)** and **(iii)** with $\hookrightarrow$-joinable critical peaks, then context-sensitive rewriting $\hookrightarrow$ is locally confluent*

Proof. A local $\hookrightarrow$-peak $s \hookleftarrow t \hookrightarrow u$ either is overlapping or not.

In the former case, the peak is an instance of a critical $\hookrightarrow$-peak occurring in some context at an active position. Then we conclude by assumption **(i)** and $\hookrightarrow$-joinability of critical peaks.

The latter case further splits into the disjoint (a) and nested redex-patterns cases (b) and (b') in the proof of [25, Lemma 2.7.15], Huet's Critical Pair Lemma. The proof of case (a) carries over directly from $\to$ to $\hookrightarrow$. The proof of cases (b) and (b') carries over as well, but using assumption **(iii)** to ensure that the residuals (at parallel positions) of the nested step remain at depth 0, so are $\hookrightarrow$-steps again. ◀

Since convectivity entails assumptions **(i)**, and $\hookrightarrow$-joinability of critical peaks and 0-preservingness entail confluence of $\hookrightarrow$ for left-linear CSRs by Lemma 24, combining this with termination of $\mathcal{T}$ all assumptions of Theorem 8 are satisfied:

▶ **Corollary 25.** *If $\mathcal{T},\mu$ is a left-linear 0-preserving CSR such that μ is convective, critical peaks are $\hookrightarrow$-joinable, and context-sensitive rewriting $\hookrightarrow$ is terminating, then the TRS $\mathcal{T}$, i.e. the rewrite system $\to$, has the Z-property for the layered bullet function $\odot$.*

This generalises [8, Theorem 2], the main result of that paper, both by *relaxing* two of its assumptions, canonicity to convectivity and level-decreasingness to 0-preservingness, and by *strengthening* its conclusion from confluence to the Z-property, in particular entailing the bullet strategy $\twoheadrightarrow_{\odot}$ is a hyper-cofinal (hence hyper-normalising) strategy [23, Lemma 51 and Theorem 50]. Moreover, the layered bullet function $\odot$ induces an *effective* (if $\hookrightarrow$ is) confluence construction and *cofinal* strategy.

▶ Remark 26. By relaxing both level-decreasingness to 0-preservingness and canonicity of the replacement map to convectivity, the corollary *partially* settles [8, Open Problem 1].

In our approach to ω-confluence in Section 3 we assumed local confluence of unrestricted rewriting $\to$ instead of of context-sensitive rewriting $\hookrightarrow$, and showed that this entails confluence in the limit, both potentially (Theorem 15; approaching the infinite normal form) and actually (Theorem 18; ω-confluence) so.

The latter result answers [8, Open Problem 2] in the affirmative, and indeed without requiring that the TRS be non-collapsing as was already suggested on [8, p. 78].

We think the former result is interesting in its own right as it stays within the world of finite terms. Comparing the conditions of our results in Sections 2 and 3, note that we even have *two* ways to establish that result for a CSR $\mathcal{T},\mu$ with $\mathcal{T}$ a left-linear TRS and μ a canonical replacement map such that $\hookrightarrow$ is terminating. On the one hand, via local confluence of $\to$ and Theorem 15 as just discussed, and on the other hand via local confluence of $\hookrightarrow$ and Corollary 25, since then, by the Z-property of $\to$ for bullet map $\odot$, the layered bullet strategy $\twoheadrightarrow_{\odot}$ is a cofinal strategy producing in each step a next layer, stable by canonicity.

▶ Remark 27. If μ is only convective, then the top layer may fail to stabilise in the layered bullet strategy $\twoheadrightarrow_{\odot}$, in that always (eventually) a $\hookrightarrow$-step may be possible. To see this, consider the CSR with rules $a \to b$ and $f(\overline{b}) \to f(\overline{a})$, which meets all the assumptions mentioned in the above except for the replacement map only being convective, not canonical (here blocking $\mu, \not\mu$-factorisation of reductions). Still, since the replacement map is convective, Corollary 25 applies, so $\twoheadrightarrow_{\odot}$ is a cofinal strategy; e.g. we have $f(\overline{a}) \twoheadrightarrow_{\odot} f(\overline{a})$ by first normalising a to b in the layer at depth 1, and next normalising $f(\overline{b})$ to $f(\overline{a})$ in the layer at depth 0.

5 Illustrating the techniques on examples

We present examples illustrating our techniques and their limitations. The examples are mostly from the literature [8, 17].

▶ **Example 28.** **(a)** The CSR with convective replacement map μ^{con} for [8, Example 1][28] is:

$$\begin{array}{rcl} g(a) & \to & f(\overline{g(a)}) \\ g(b) & \to & c \\ a & \to & b \\ f(\overline{x}) & \to & h(\overline{x},\overline{x}) \\ h(\overline{x},\overline{y}) & \to & c \end{array}$$

Due to the critical peak between the first and third rules, convectivity entails we must at least have $1 \in \mu(g)$. Since with this convective replacement map μ^{con} the critical peak can be completed to a $\hookrightarrow$-diagram with legs $g(a) \hookrightarrow f(\overline{g(a)}) \hookrightarrow h(\overline{g(a)},\overline{g(a)}) \hookrightarrow c$ and $g(a) \hookrightarrow g(b) \hookrightarrow c$ and the rules are vacuously 0-preserving in the absence of variables occurring at depth 0 in left-hand sides, Corollary 25 applies so $\to$ has the Z-property, is confluent, and $-\!\!\odot\!\!\to$ is a cofinal $\twoheadrightarrow$-strategy.

Since in this case the convective replacement map μ^{con} is the same as the canonical replacement map μ^{can} of [8, Example 1] and the rules are seen to be level-decreasing (all occurrences of the variables are at depth 1) confluence of $\to$ is in this case also a consequence of the main result of [8] as was observed on [8, p. 75].

By canonicity also Theorems 15 and 18 apply to yield ω-confluence.

(b) The CSR with convective replacement map μ^{con} for [8, Example 2][29] is:

$$\begin{array}{rcl} \mathsf{nats} & \to & \overline{0} : \overline{\mathsf{inc}(\mathsf{nats})} \\ \mathsf{inc}(\overline{x} : \overline{y}) & \to & \mathsf{s}(\overline{x}) : \overline{\mathsf{inc}(y)} \\ \mathsf{hd}(\overline{\overline{x} : \overline{y}}) & \to & x \\ \mathsf{tl}(\overline{x} : \overline{y}) & \to & y \\ \mathsf{inc}(\mathsf{tl}(\mathsf{nats})) & \to & \mathsf{tl}(\mathsf{inc}(\mathsf{nats})) \end{array}$$

Due to the critical peak between the first and fifth rules, convectivity entails we must at least have $1 \in \mu(\mathsf{inc}), \mu(\mathsf{tl})$. Since with this convective replacement map μ^{con} the critical peak can be completed to a $\hookrightarrow$-diagram with legs $\mathsf{inc}(\mathsf{tl}(\mathsf{nats})) \hookrightarrow \mathsf{tl}(\mathsf{inc}(\mathsf{nats})) \hookrightarrow \mathsf{tl}(\mathsf{inc}(\overline{0}{:}\overline{\mathsf{inc}(\mathsf{nats})})) \hookrightarrow \mathsf{tl}(\mathsf{s}(\overline{0}){:}\overline{\mathsf{inc}(\mathsf{inc}(\mathsf{nats}))}) \hookrightarrow \mathsf{inc}(\mathsf{inc}(\mathsf{nats}))$ and $\mathsf{inc}(\mathsf{tl}(\mathsf{nats})) \hookrightarrow \mathsf{inc}(\mathsf{tl}(\overline{0}{:}\overline{\mathsf{inc}(\mathsf{nats})})) \hookrightarrow \mathsf{inc}(\mathsf{inc}(\mathsf{nats}))$ and all rules are vacuously 0-preserving in the absence of variables occurring at depth 0 in left-hand sides, Corollary 25 applies so $\to$ has the Z-property, is confluent, and $-\!\!\odot\!\!\to$ is a cofinal $\twoheadrightarrow$-strategy.

In this case the convective replacement map μ^{con} is not the same as the canonical replacement map μ^{can}:[30] since in the left-hand side of the third rule the argument of hd has the function symbol : as head symbol, canonicity requires that $1 \in \mu^{can}(\mathsf{hd})$. This results in the canonical replacement map μ^{can} and with this the critical peak and its diagram remain as above (hd does not occur in it). However, to obtain confluence of $\to$ as a *consequence* of the main result of [8], rules also need to be level-decreasing. For the

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[28] The TRS is COPS #19. The claim there that this is Example **2** of [8, Example 1] seems a typo?

[29] Methods to prove confluence of this TRS (COPS #20) are the theme of [10].

[30] Although our μ^{con} coincides with the replacement map given on [8, p. 70] for these rules, the claim there that that *is* the canonical replacement map μ^{can} cannot be correct I think, for the reason given above.

second rule this also entails $1 \in \mu(\mathsf{s})$. This replacement map works (note that critical peak diagrams of $\hookrightarrow$ are preserved by making the replacement map less restrictive, in this case by changing the argument of s from frozen into active).
By canonicity of the resulting replacement map, not only their [8, Theorem 2] applies to yield confluence, but also our Theorems 15 and 18 apply to yield ω-confluence.

(c) Consider the CSR[31] obtained by replacing the first and last rule in the previous item by the following three rules and preserving the replacement map:

$$\begin{aligned} \mathsf{nats} &\to \mathsf{from}(\overline{0}) \\ \mathsf{from}(\overline{x}) &\to \overline{x} : \overline{\mathsf{from}(\overline{\mathsf{s}(\overline{x})})} \\ \mathsf{inc}(\mathsf{tl}(\mathsf{from}(\overline{x}))) &\to \mathsf{tl}(\mathsf{inc}(\mathsf{from}(\overline{x}))) \end{aligned}$$

Due to the critical peak between between the second and third added rules, also for these rules that replacement map is the convective replacement map μ^{con}. That critical peak gives rise to the diagram with legs $\mathsf{inc}(\mathsf{tl}(\mathsf{from}(\overline{x}))) \hookrightarrow \mathsf{tl}(\mathsf{inc}(\mathsf{from}(\overline{x}))) \hookrightarrow \mathsf{tl}(\mathsf{inc}(\overline{x} : \overline{\mathsf{from}(\overline{\mathsf{s}(\overline{x})})})) \hookrightarrow \mathsf{tl}(\overline{\mathsf{s}(\overline{x})} : \overline{\mathsf{inc}(\mathsf{from}(\overline{\mathsf{s}(\overline{x})}))}) \hookrightarrow \mathsf{inc}(\mathsf{from}(\overline{\mathsf{s}(\overline{x})}))$ and $\mathsf{inc}(\mathsf{tl}(\mathsf{from}(\overline{x}))) \hookrightarrow \mathsf{inc}(\mathsf{tl}(\overline{x} : \overline{\mathsf{from}(\overline{\mathsf{s}(\overline{x})})})) \hookrightarrow \mathsf{inc}(\mathsf{from}(\overline{\mathsf{s}(\overline{x})}))$ and since rules are still vacuously 0-preserving, Corollary 25 applies so $\to$ has the Z-property, is confluent, and $\multimap\!\!\!\bullet\!\!\!\to$ is a cofinal $\twoheadrightarrow$-strategy.
As in the previous item, canonicity requires we have $1 \in \mu^{can}(\mathsf{hd})$. The resulting replacement map is μ^{can} hence is convective and since the rules are still 0-preserving Corollary 25 still applies with consequences as before, in particular that $\to$ is confluent.
However, this time that cannot be obtained by the methods of [8]. These require level-decreasingness of the rules and the second added rule is not for μ^{can}: the level of x in the lhs is 1 whereas in the rhs it occurs not only with level 1 but also with level 3. The only way to regain level-decreasingness is to make both the second argument of : and the argument of s accessible, but that would violate termination of $\hookrightarrow$ (the second added rule becomes spiralling), one of the other assumptions of [8, Theorem 2].
However, for our Theorems 15 and 18 level-decreasingness is irrelevant, canonicity and termination of $\hookrightarrow$ suffice, yielding also ω-confluence of $\to$.

(d) A CSR in a spirit similar to those in the previous two items is obtained from the TRS in [25, Section 12.1], used there as a motivating example for infinitary rewriting, with the convective replacement map μ^{con}, which is empty here since the TRS is orthogonal:

$$\begin{aligned} \mathsf{filter}(\overline{\overline{x} : \overline{y}}, \overline{0}, \overline{m}) &\to \overline{0} : \overline{\mathsf{filter}(\overline{y}, \overline{m}, \overline{m})} \\ \mathsf{filter}(\overline{\overline{x} : \overline{y}}, \overline{\mathsf{s}(\overline{n})}, \overline{m}) &\to \overline{x} : \overline{\mathsf{filter}(\overline{y}, \overline{n}, \overline{m})} \\ \mathsf{sieve}(\overline{\overline{0} : \overline{y}}) &\to \mathsf{sieve}(\overline{y}) \\ \mathsf{sieve}(\overline{\overline{\mathsf{s}(\overline{n})} : \overline{y}}) &\to \overline{\mathsf{s}(\overline{n})} : \overline{\mathsf{sieve}(\overline{\mathsf{filter}(\overline{y}, \overline{n}, \overline{n})})} \\ \mathsf{nats}(\overline{n}) &\to \overline{n} : \overline{\mathsf{nats}(\overline{\mathsf{s}(\overline{n})})} \\ \mathsf{primes} &\to \mathsf{sieve}(\overline{\mathsf{nats}(\overline{\mathsf{s}(\overline{\mathsf{s}(\overline{0})})})}) \end{aligned}$$

Since $\hookrightarrow$ has no critical peaks and is trivially terminating (only $\hookrightarrow$-redexes for the third and fourth rules can be created, with the former being size-decreasing and the latter yielding a $\hookrightarrow$-normal form), Corollary 25 applies so $\to$ has the Z-property, is confluent, and $\multimap\!\!\!\bullet\!\!\!\to$ is a cofinal $\twoheadrightarrow$-strategy.

[31] Suggested to us by Nao Hirokawa as an example of a system that can be handled by our context-sensitive methods but not by those of [8].

For the canonical replacement map μ^{can} has $1, 2 \in \mu^{can}(\mathsf{filter})$, $1 \in \mu^{can}(\mathsf{sieve})$ and $1 \in \mu^{can}(:)$, but $\hookrightarrow$ is not terminating[32] since $\mathsf{sieve}(\mathsf{filter}(\mathsf{nats}(\overline{n}), 0, \overline{0})) \hookrightarrow \mathsf{sieve}(\mathsf{filter}(n : \overline{\mathsf{nats}(\overline{\mathsf{s}(\overline{n})})}, 0, \overline{0})) \hookrightarrow \mathsf{sieve}(0 : \overline{\mathsf{filter}(\mathsf{nats}(\overline{\mathsf{s}(\overline{n})}), 0, \overline{0})}) \hookrightarrow \mathsf{sieve}(\mathsf{filter}(\mathsf{nats}(\overline{\mathsf{s}(\overline{n})}), 0, \overline{0}))$ giving rise to an infinite spiralling reduction. Since for all canonical replacement maps $\to$ is 'less' terminating than for μ^{can}, the methods of [8] cannot be applied to yield confluence of this example, and for the same reason neither can ω-confluence be shown by our Theorem 15.

Just like it is interesting to restrict termination to *basic* terms, function(symbol)s applied to terms comprising constructors only, it is interesting to restrict productivity to basic terms / a given basic term. Given that this TRS was designed to generate, starting from primes, the infinite list of prime numbers, it is no surprise that that infinite list is *produced* by the context-free $\hookrightarrow$-strategy, and one expects both confluence and ω-confluence to hold for basic terms. At the same time, to show productivity for primes obviously requires Euclid's result that there are infinitely many prime numbers, so should be challenging to establish automatically (note that primes would *not* be productive if we were to replace its rhs by $\mathsf{sieve}(\overline{\mathsf{nats}(\overline{\mathsf{s}(\overline{0})})})$, i.e. by simply removing an s).

(e) A CSR with convective replacement map μ for [8, Example 3] is:

$$\begin{array}{lcl} b & \to & a \\ b & \to & c \\ c & \to & h(\overline{b}) \\ c & \to & d \\ a & \to & h(\overline{a}) \\ d & \to & h(\overline{d}) \end{array}$$

For this CSR $\hookrightarrow$ is obviously not confluent for the critical peak $a \hookleftarrow b \hookrightarrow c$: the respective $\hookrightarrow$-reduction graphs $a \hookrightarrow h(\overline{a})$ of a and $c \hookrightarrow h(\overline{b})$, $c \hookrightarrow d \hookrightarrow h(\overline{d})$ of c are disjoint. For the other convective replacement map, μ^{con}, with the only difference being that the argument of h is active, $\hookrightarrow$ is obviously not terminating (the fifth rule then is spiralling). Hence our results do not apply to yield confluence of $\to$. This is as it should be: since $\to$ is *not* confluent, they should not apply [8, p. 75].

Still since the replacement map μ is canonical, $\hookrightarrow$ is terminating (the argument of h being frozen blocks spirals / non-termination), and $\to$ *is* locally confluent [8, Example 3], the assumptions of Theorems 15 and 18 *are* satisfied, yielding ω-confluence. For instance, $t := h(\overline{t})$ is the unique infinitary normal form of b, independent of whether we reduce b to, say, a or c first.

Note that the road to ω-confluence via *any* cofinal strategy (not just cofinal strategies induced by the Z-property) is blocked here, simply because cofinality would entail confluence, and $\to$ is not confluent.

(f) Consider the following CSR, a modification of that in the first item, for the TRS of [8, Example 5]:

$$\begin{array}{rcl} g(a) & \to & f(\overline{g(a)}) \\ g(b) & \to & c(\overline{a}) \\ a & \to & b \\ f(\overline{x}) & \to & h(\overline{x}) \\ h(\overline{x}) & \to & c(\overline{b}) \end{array}$$

[32] As found automatically by, e.g., Aprove.

Due to the critical peak between the first and third rules we must have $1 \in \mu(g)$ for replacement map μ. As observed in [8, Example 5], the critical peak is not $\hookrightarrow$-joinable for this μ: $g(a) \hookrightarrow f(\overline{g(a)}) \hookrightarrow h(\overline{g(a)}) \hookrightarrow c(\overline{b})$ and $g(a) \hookrightarrow g(b) \hookrightarrow c(\overline{a})$. In a CSR such a non-confluence peak can be *completed* in *two* ways, either in the classical way of adjoining a rule between the respective targets (in $\hookrightarrow$-normal form) $c(\overline{a}) \hookrightarrow c(\overline{b})$, or by making the argument of c active (since we already did have $c(\overline{a}) \to c(\overline{b})$). However, the former way gives rise to a new critical peak with the third rule that is not a critical peak of the CSR, so to which our results do not apply. As observed in [8, Example 5] the latter way does work however, preserving termination of $\hookrightarrow$ and completing the diagram by the step $c(a) \hookrightarrow c(b)$, yielding a CSR to which [8, Theorem 2] applies so $\to$ is confluent, hence also Corollary 25 applies so $\to$ has the Z-property, is confluent, and $\mathrel{-\!\!\bullet\!\!\rightarrow}$ is a cofinal $\twoheadrightarrow$-strategy, and Theorems 15 and 18 apply to yield ω-confluence.

(g) Combining the TRS of [8, Example 6] with the convective replacement map μ^{con}, which is the empty replacement here by orthogonality of the TRS, yields the CSR:

$$\begin{array}{rcl} \mathsf{from}(\overline{x}) & \to & \overline{x} : \overline{\mathsf{from}(\overline{\mathsf{s}(\overline{x})})} \\ \mathsf{sel}(\overline{0}, \overline{\overline{y} : \overline{z}}) & \to & y \\ \mathsf{sel}(\overline{\mathsf{s}(\overline{x})}, \overline{\overline{y} : \overline{z}}) & \to & \mathsf{sel}(\overline{x}, \overline{z}) \end{array}$$

As argued in Remark 2**(c)**, for applicability Corollary 25 it suffices to check context-sensitive rewriting $\hookrightarrow$ is terminating, It trivially is (only sel-steps are of interest and these are size-decreasing), so $\to$ has the Z-property by Corollary 25.
Since context-sensitive rewriting $\hookrightarrow$ is also terminating for replacement map μ^{can} [8, Example 6], Theorems 15 and 18 apply to yield ω-confluence.
This example served in [8] to exemplify the *limitations* of the results presented there; their methods do *not* apply to this TRS, they fail to show its confluence. In particular, for a replacement map to be level-decreasing as required by them, the second argument of : must be active, entailing non-termination of $\hookrightarrow$ (as in the third item).

(h) For the CSR $\mathcal{T}, \mu$ with rules

$$\begin{array}{rcl} f(x) & \to & c(\overline{f(x)}) \\ f(x) & \to & c(\overline{f(f(x))}) \end{array}$$

the TRS is left-linear and ($\to$ is) confluent (as e.g. shown by decreasing diagrams and rule labelling), μ is canonical (and convective), and context sensitive rewriting $\hookrightarrow$ is terminating. The context free $\hookrightarrow$-strategy (layered CSR) therefore is a hyper-normalising $\to$-strategy and hence the parallel-outermost strategy $\mathrel{+\!\!\!\!\rightarrow}_{\mathsf{po}}$ is so as well, as shown in [10]. However these results are not relevant here since terms in this CSR typically do not have a normal form. Still, every such term does have an infinite normal form, $t := c(\overline{t})$, and Remark 4**(e)** yields that that is found by either strategy. (Fairness is not an issue here since there are no binary function symbols, but note that even when adjoining such, $\mathrel{+\!\!\!\!\rightarrow}_{\mathsf{po}}$ remains infinitary hyper-normalising since it is fair by maximality.) That (infinite) normal forms are unique follows from ω-confluence, which holds by Theorem 15.
Note that $\hookrightarrow$ is not confluent since the peak $c(\overline{f(x)}) \hookleftarrow f(x) \hookrightarrow c(\overline{f(f(x))})$ is not $\hookrightarrow$-joinable, so the route to infinitary normalisation via the Z-property, to yield cofinality of the bullet strategy $\mathrel{-\!\!\bullet\!\!\rightarrow}$, is blocked; Theorem 8 is not applicable. Even stronger, neither the parallel outermost strategy $\mathrel{+\!\!\!\!\rightarrow}_{\mathsf{po}}$ of [10] nor the bullet strategy $\mathrel{-\!\!\bullet\!\!\rightarrow}$ are in fact cofinal in this case. Starting from the term $f(x)$ both strategies may give rise to an infinite reduction with successive terms $f(x), c(\overline{f(x)}), c(\overline{c(\overline{f(x)})}), \ldots$ by always selecting the first

rule,[33] and *no* term in this sequence is reachable from the target $c(\overline{f(f(x))})$ of the step $f(x) \to c(\overline{f(f(x))})$ (they would need to have at least two occurrences of f).[34]

(i) Consider the TRS $\mathcal{T}$ on [8, p. 78] with rules:[35]

$$\begin{array}{rcl} a(x) & \to & x \\ b(x) & \to & x \\ c & \to & a(b(c)) \end{array}$$

Since there is the head loop $c \hookrightarrow a(b(c)) \hookrightarrow b(c) \hookrightarrow c$, for any CSR $\mathcal{T}, \mu$ context-free rewriting $\hookrightarrow$ is non-terminating independent of the replacement map μ. Hence none of our techniques apply to this example.

Although the full development bullet map $\bullet$ of [23, Definition 19] has the Z-property since $\mathcal{T}$ is orthogonal, so $\multimap\!\!\!\!\bullet\!\!\rightarrow$ is a cofinal strategy,[36] that is of no avail here because c has *no* (head-)normal form as a consequence of that it is $\twoheadrightarrow$*–recurrent* in the sense of Statman: if $c \twoheadrightarrow t$ then $t \twoheadrightarrow c$ [23, Definition 54]. Interestingly, this can be shown by the very same Z-property, since both $c \multimap\!\!\!\!\bullet\!\!\rightarrow a(b(c))$ and $a(b(c)) \twoheadrightarrow c$, see [23, Remark 53].

(j) Consider the CSR of [8, p. 78] (a variation on the CSR in the previous item) with rules:

$$\begin{array}{rcl} a(x) & \to & x \\ b(x) & \to & x \\ c & \to & d(\overline{a(b(c))}) \end{array}$$

To this CSR all our techniques apply yielding the Z-property and ω-confluence.

(k) The CSR of [17, Example 38]:

$$\begin{array}{rcl} f(x) & \to & g(\overline{x}) \\ g(\overline{x}) & \to & x \end{array}$$

is not 0-preserving (due to the first rule), so Corollary 25 does not apply. Still, since $\hookrightarrow$ *is* confluent (as argued in [17, Example 53] based on the extended-critical-pair results developed there, delegating some proof obligations to Prover9), Theorem 8 does apply, so the Z-property holds. The CSR also satisfies the assumptions of Theorem 15 so ω-confluence holds. (Note these consequences are obvious anyway by orthogonality and termination of the TRS).

(l) There is no critical peak between the first two rules of the CSR:

$$\begin{array}{rcl} a & \to & b \\ f(\overline{a}) & \to & c \\ c & \to & f(\overline{b}) \end{array}$$

though there is a critical peak in the TRS hence the indicated replacement map is not convective. Hence our methods do not apply to this CSR, though the TRS is confluent as noted in Remark 2**(b)**. For the only other replacement map, which is equal to both μ^{can} and μ^{con}, all our methods apply, yielding both confluence and ω-confluence, which however is neither helpful (the CSR *is* the TRS) nor surprising (the TRS is trivially shown to be complete).

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[33] The problem with $\multimap\!\!\!\!\odot\!\!\rightarrow$ is thus that it is not even well-defined since $\bullet$ is not, by non-confluence of $\hookrightarrow$.

[34] Always contracting the outermost redex by the second rule does seem to be a cofinal strategy here.

[35] This is the canonical example due to Kennaway, cf. [25, Chapter 12] of an orthogonal (hence confluent) TRS that is not ω-confluent, since both $A := a(A)$ and $B := b(B)$ are reachable from c.

[36] This strategy is also known as *Gross–Knuth* reduction or *full substitution*.

(m) Context-sensitive rewriting $\hookrightarrow$ is trivially terminating for the left-linear and canonical CSR:

$$\begin{array}{rcl} a & \to & b \\ a & \to & s(\overline{b}) \\ b & \to & s(\overline{s(\overline{b})}) \end{array}$$

Its critical peak $b \hookleftarrow a \hookrightarrow s(b)$ is not $\to$-joinable (the numbers of bs in the reducts of its targets b and $s(\overline{b})$ are even and odd respectively; they are out-of-sync), so neither $\hookrightarrow$ nor $\to$ is confluent, and Theorem 18 cannot be applied to yield ω-confluence. However, the targets b and $s(\overline{b})$ of the critical peak reduce to the terms $s(\overline{s(\overline{b})})$ and $s(\overline{b})$ having the same layer $s(\overline{\Box})$ at depth 0, and arguments $s(\overline{b})$ and b at depth 1 that are convertible by $s(\overline{b}) \leftarrow a \to b$, hence 0-local confluence holds for the (only) critical $\hookrightarrow$-peak. Since the rules are vacuously non-0-collapsing, we conclude by Lemma 21 that 0-local confluence holds for all $\hookrightarrow$-peaks, so we may conclude to ω-confluence by Corollary 20.

(n) Consider the weakly orthogonal CSR:

$$\begin{array}{rcl} d & \to & b(0,0) \\ a(0,m) & \to & b(m,\overline{r(m)}) \\ a(r(n),m) & \to & s(\overline{a(n,m)}) \\ b(0,m) & \to & a(m,\overline{r(m)}) \\ b(r(n),m) & \to & p(\overline{b(n,m)}) \\ p(\overline{s(\overline{x})}) & \to & x \\ s(\overline{p(\overline{x})}) & \to & x \end{array}$$

Since the CSR is not even convective (due to the critical peaks between the last two rules), none of our techniques applies. Note that although the TRS is weakly orthogonal so confluent on finite terms, the CSR is not ω-confluent: from d there are strongly converging ω-reductions to the distinct infinite normal forms s^ω and p^ω. Thus, that our results for ω-confluence do not apply to it, is as it should be.

Our methods still do not apply after making the arguments of p and s active to restore convectivity, as that leads to failure of termination of $\hookrightarrow$, which can be seen e.g. by that d then produces $p(s(s(p(p(p(\ldots))))))$ as argued above.

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