The Z-property for left-linear term rewriting via convective context-sensitive completeness*

Vincent van Oostrom

oostrom@javakade.nl

Abstract

We present a method to derive the Z-property, hence confluence, of a first-order term rewrite system $\mathcal{T}$ from completeness of an associated context-sensitive term rewrite system $\mathcal{T},\mu$ with replacement map μ. We generalise earlier such results by only requiring left-linearity of $\mathcal{T}$ and that $\mathcal{T}$-critical peaks be $\mathcal{T},\mu$-critical peaks. We introduce convective replacement maps as a generalisation of canonical maps known from the literature.

Background The direct inspiration for this note was the invited IWC 2022 presentation [2], in particular its contemplation of *cofinal* strategies [8], which raised the obvious question whether the Z-property could play a rôle in the theory developed (by Hirokawa based on earlier work of Gramlich and Lucas), as it is known that the Z-property gives rise to a (hyper-)cofinal *bullet* strategy [6], and entails confluence. We answer that question in the affirmative.

More in particular, this note concerns a method to *transfer* confluence of a terminating context-sensitive term rewrite system (CSR) $\mathcal{T},\mu$ to its underlying term rewrite system (TRS) $\mathcal{T}$. We provide two assumptions allowing to establish the Z-property [6] for a TRS and its *layered bullet map* $\circledbullet$, introduced here, that inside–out and layer-wise $\mathcal{T},\mu$-normalises a term, where the notion of layer is afforded by the replacement map μ of the CSR.

Preliminaries. For first-order term rewriting we base ourselves on [8], for context-sensitive term rewriting on [1], and for the Z-property on [6], with which we assume the reader has a nodding familiarity. Though we will recapitulate some key notions relevant to the developments here, we refer the reader to that literature for background information.

Context-sensitive term rewrite systems are term rewrite systems, where contracting a redex is restricted by a so-called *replacement map* mapping each function symbol in the signature to its set of *active* argument positions. The notion of being active extends compositionally to an occurrence of one term in another, via the latter occurring only in active arguments of the function symbols occurring on its path from the root in the former. Given a replacement map, context-sensitive rewriting only allows to contract active occurrences of redexes. Formally, for μ a replacement map, a *μ-redex* is a redex at an active occurrence.

Given a context-sensitive term rewrite system (CSR) $\mathcal{T},\mu$, with $\mathcal{T}$ a term rewrite system (TRS) and μ a replacement map μ, we use $\to$ to denote the rewrite system induced by $\mathcal{T}$, and $\hookrightarrow$ to denote the rewrite system induced by $\mathcal{T},\mu$, contracting μ-redexes only.

Remark 1. Ordinary term rewriting is the special case of context-sensitive term rewriting, via the replacement map in which *all* arguments of *all* function symbols are active.

 It's a some–typos–corrected and slightly–reformatted version of a note submitted (28-5-2023) to IWC 2023. It is derived from http://www.javakade.nl/research/pdf/z-csr.pdf. Updated on 5-6-2023 with Appendix A.

We will exploit that, despite appearances, whether or not the *occurrence*[1] $\langle t \mid C[\,]\rangle$ of one term t in another $s = C[t]$ is active, does not depend on the (whole) *context* $C[\,]$, but only on the function symbols occurring on its *access path*, the path from the root to the hole of the context.

The main technique. We are interested in transferring confluence of $\hookrightarrow$ to that of $\to$. To that end, we will work throughout under the following two assumptions.

(i) $\mathcal{T}$ critical peaks are $\mathcal{T},\mu$ critical peaks.

(ii) $\mathcal{T},\mu$ is a left-linear and complete (confluent and terminating) CSR.

Remark 2. (1) Without assumption **(i)** one can't expect to *transfer* confluence from $\hookrightarrow$ to $\to$, simply because context-sensitive rewriting in $\mathcal{T},\mu$ may miss out on (say nothing about) critical peaks of $\mathcal{T}$. For instance, consider the TRS $\mathcal{T}$ with rules $a \to b$ and $f(\overline{a}) \to c$ where we used (as we will do below) overlining[2] to indicate that the argument of f is frozen, i.e. that $\mu(f) := \emptyset$. Then $\hookrightarrow$ is confluent, which may be shown by checking that the only $\hookrightarrow$-reducible terms are a and $f(\overline{a})$, and those are deterministic. In particular, we do *not* have $f(\overline{a}) \hookrightarrow f(\overline{b})$ since a is frozen in $f(\overline{a})$, see [1, 5]. However, $\to$ is not confluent due to the non-joinable critical peak $f(\overline{b}) \leftarrow f(\overline{a}) \hookrightarrow c$. (2) Neither assumption **(i)** nor assumption **(ii)** is necessary. That assumption **(i)** is not, may be shown by adjoining $c \to f(\overline{b})$ to $\mathcal{T}$. That preserves confluence of $\hookrightarrow$, which may be transferred to confluence of $\to$ using that the source of $f(\overline{a}) \to f(\overline{b})$ is $\hookrightarrow$-reducible to its target: $f(\overline{a}) \hookrightarrow c \hookrightarrow f(\overline{b})$, showing that the problematic critical peak is *redundant*, cf. [3].

To *maximise* the chance that the context-sensitive rewrite system $\hookrightarrow$ is terminating, i.e. to maximise applicability of assumption **(ii)**, it is best to *minimise* the number of active arguments or, stated differently, to *maximise* the number of frozen arguments [1]. That is, letting μ map each function symbol to the empty set $\emptyset$ would be best, but that may not be possible as assumption **(i)** forces for every rule $\ell \to r$ that for every position p in ℓ such that $\ell|_p$ unifies with some left-hand side of a rule, p be active / not frozen. This motivates:

Definition 3 (convective)**.** A replacement map μ is *convective* if $\mu^{cnv} \subseteq \mu$, i.e. if μ is not more restrictive than μ^{cnv}, where μ^{cnv} is the most restrictive replacement map such that for every rule $\ell \to r$, for every position p in ℓ such that $\ell|_p$ unifies with some left-hand side of a rule (i.e. an overlap), $i \in \mu^{cnv}(\ell(q))$ for any $qi \preceq p$ (i.e. q is the position of a function symbol on the path from the root to the overlap position p and i is its argument for which this holds.

Convectivity guarantees that if two left-hand sides occurring in a term have overlap the one is active iff the other is, but nothing more. In particular, in a critical peak the inner redex occurrence is active since the outer occurrence, at the root, is.

Example 4 (convective running example)**.** *Consider the CSR*[3] *having rules and replacement map* μ^{cnv}*:*

$$\begin{array}{rcl}
\mathsf{nats} & \to & \mathsf{from}(\overline{0}) \\
\mathsf{inc}(\overline{x} : \overline{y}) & \to & \overline{\mathsf{s}(\overline{x})} : \overline{\mathsf{inc}(y)} \\
\mathsf{hd}(\overline{\overline{x} : \overline{y}}) & \to & x \\
\mathsf{tl}(\overline{x} : \overline{y}) & \to & y \\
\mathsf{from}(\overline{x}) & \to & \overline{\overline{x} : \mathsf{from}(\overline{\mathsf{s}(\overline{x})})} \\
\mathsf{inc}(\mathsf{tl}(\mathsf{from}(\overline{x}))) & \to & \mathsf{tl}(\mathsf{inc}(\mathsf{from}(\overline{x})))
\end{array}$$

[1] To be formal here we use the not(at)ion of occurrence of [8, Sect. 2.1.1]. Below we will make do with the usual informal ways of specifying occurrences.

[2] Our overlining notation suggests that the overlined argument position is *cut off* from its context, i.e. *frozen*.

[3] Suggested to us by Nao Hirokawa.

The only critical peak is between the fifth and sixth rules, for which convectivity entails we must at least have $1 \in \mu(\mathsf{inc}), \mu(\mathsf{tl})$. *These two constraints give the convective replacement map* μ^{cnv}.

For this CSR $\mathcal{T}, \mu$ *context-sensitive rewriting* $\hookrightarrow$ *trivially (easily checked by tools) is terminating, whereas ordinary term rewriting for* $\mathcal{T}$ *trivially is non-terminating.*

Remark 5. In the literature so-called *canonical* replacement maps, for which only the variables may occur frozen in the left-hand sides of rewrite rules, play an important rôle. Formally, μ is *canonical* if $\mu^{can} \subseteq \mu$, i.e. if μ is not more *restrictive* than μ^{can}, where μ^{can} is defined by $i \in \mu^{can}(f)$ if for some position p and some rule $\ell \to r$, we have $\ell(p) = f$ and $\ell(pi)$ is a function symbol.

Following-up on the preliminaries, the intuitive difference between canonical and convective replacement maps is that a canonical replacement map requires *all* (non-variable) positions in the redex-pattern of a redex to be active, whereas a convective replacement map requires this only of the positions in a redex-pattern on an *access path* to where it may be overlapped by another redex.

Example 6. *In Ex. 4 canonicity requires we also have* $1 \in \mu^{can}(\mathsf{hd})$ *due to the (first) argument belonging to the pattern of the left-hand side of the third rule, illustrating* $\mu^{cnv} \subset \mu^{can}$ *here.*

The idea of our terminology *convective* is to view a term as a fluid, and the paths from the root of a left-hand side to the roots of overlapping left-hand sides as representing flows within the fluid, with the flow enabling *activation* of the latter. A term is in $\hookrightarrow$-normal form iff there's no *flow* from the root of the term to any redex-pattern. It then makes some intuitive sense to speak of its layer at depth 0 as being *solid*. Formally, the *depth* of an occurrence is the number of frozen argument positions it is in on the path to the root, inducing a natural stratification of terms into *layers* of symbols, subterms, and redexes occurring at a given depth.

Lemma 7. *If* $t \to s$ *then* $t^\bullet \twoheadrightarrow s^\bullet$, *where* $\bullet$ *maps a term to its* $\hookrightarrow$*-normal form, existing uniquely by assumption* ***(ii)***.[4]

Proof of Lem. 7. We claim $t \mathrel{+\!\!\!\!\!\rightarrow} s$ entails[5] $t^\bullet \twoheadrightarrow \hat{s} \twoheadleftarrow\!\!\hookleftarrow s$ for some $\hat{s}$. From the claim we conclude using $\hat{s} \twoheadrightarrow s^\bullet$ by assumption **(ii)** and $\hookrightarrow \subseteq \to$. We prove the claim by induction on t w.r.t. $\hookleftarrow$ well-founded[6] by assumption **(ii)**, and by distinguishing cases on $t \mathrel{+\!\!\!\!\!\rightarrow} s$:

If $t \mathrel{+\!\!\!\!\!\rightarrow} s$ decomposes as $t \hookrightarrow t' \mathrel{+\!\!\!\!\!\rightarrow} s$, we conclude by the IH for $t' \mathrel{+\!\!\!\!\!\rightarrow} s$ and $t^\bullet = t'^\bullet$.

Otherwise $t \mathrel{+\!\!\!\!\!\rightarrow} s$ only contracts non-μ-redexes, occurring at depths at least 1 in t. By assumption **(i)** those cannot have overlap with any redex-pattern at depth 0 in t, as that would give rise to a critical peak of $\mathcal{T}$ that is not a critical peak of $\mathcal{T}, \mu$.

If $t = t^\bullet$ we may trivially set $\hat{s} := s$.

Otherwise, for some t' there is a step $t \hookrightarrow t'$ orthogonal to $t \mathrel{+\!\!\!\!\!\rightarrow} s$, hence by the assumed left-linearity of $\mathcal{T}$ the steps commute. Because $t \hookrightarrow t'$ is not below (any redex-pattern in) $t \mathrel{+\!\!\!\!\!\rightarrow} s$, the residual of the former after the latter is again a (single) $\hookrightarrow$-step, inducing a diagram of shape $t \hookrightarrow t' \mathrel{+\!\!\!\!\!\rightarrow} s' \hookleftarrow s$. By the IH for $t' \mathrel{+\!\!\!\!\!\rightarrow} s'$ and assumption **(ii)** we conclude to $t^\bullet = t'^\bullet \twoheadrightarrow \hat{s} \twoheadleftarrow\!\!\hookleftarrow s' \hookleftarrow s$ for some $\hat{s}$, as desired. ☐

[4]We employ Klop's convention, cf. [8], to use an arrow with a *double* arrowhead to denote the *reflexive–transitive* closure of the rewrite relation denoted by the arrow with a *single* arrowhead.

[5]We employ Huet's convention, cf. [8], to use an arrow adorned with two vertical strokes to denote *parallel* reduction, allowing to perform steps with respect to the unadorned reduction at an arbitrary number of *parallel* positions in parallel.

[6]Recall the widespread convention (used in proof theory, in rewriting [8], in proof assistants (Coq)) to say a relation $\hookleftarrow$ is *well-founded* if there are no infinite *descending* sequences $\ldots \hookleftarrow a_2 \hookleftarrow a_1 \hookleftarrow a_0$.

Assumption **(ii)** ensures $\hookrightarrow$ has the Z-property[7] for *bullet* map $\bullet$ by [6, Lem. 11]. That bullet map is *extensive* for $\hookrightarrow$, i.e. $t \hookrightarrow\!\!\!\!\rightarrow t^\bullet$ [6, Definition 4]. We show $\to$ has the Z-property under assumptions **(i)** and **(ii)** for some bullet map $\circledbullet$ based on $\bullet$. To define $\circledbullet$ we use that any term can be uniquely decomposed into its *active* layer at depth 0 w.r.t. μ,[8] and its *frozen* arguments at depth 1. Accordingly, we write $C\langle\vec{t}\rangle$ to denote such a unique decomposition, where C is the active layer and $\vec{t}$ the (vector of) frozen arguments.

Definition 8. The *layering* $\circledbullet$ (of $\bullet$) is inductively defined by $C\langle\vec{t}\rangle^{\circledbullet} := C\langle\vec{t^{\circledbullet}}\rangle^\bullet$.

Lemma 9. $C[\vec{t^{\circledbullet}}] \twoheadrightarrow C[\vec{t}]^{\circledbullet}$

Proof. By induction and cases on C. The base cases $C = \Box$ and $C = x$ being trivial, suppose C has shape $f(\vec{C})$ and decompose $\vec{t}$ accordingly. We conclude to $C[\vec{t^{\circledbullet}}] = f(\overrightarrow{C[\vec{t^{\circledbullet}}]}) \twoheadrightarrow f(\overrightarrow{C[\vec{t}]^{\circledbullet}}) \twoheadrightarrow f(\overrightarrow{C[\vec{t}]})^{\circledbullet} = C[\vec{t}]^{\circledbullet}$ by, respectively, the decomposition of $C[\vec{t}]$, the induction hypothesis for $\vec{C}$ and closure under contexts of $\to$, the claim that $g(\vec{s^{\circledbullet}}) \twoheadrightarrow g(\vec{s})^{\circledbullet}$ for all g and $\vec{s}$, and by definition of the decomposition again.

To prove the claim, first observe that $g(\vec{s^{\circledbullet}}) \twoheadrightarrow g(\vec{s^{\circledbullet}})^\bullet$ by extensivity of $\bullet$ and $\hookrightarrow \subseteq \to$. Therefore, to conclude it suffices to show $g(\vec{s^{\circledbullet}})^\bullet = g(\vec{s})^{\circledbullet}$. To that end, let $g(\vec{s})$ uniquely decompose as $g(\overrightarrow{D[\vec{u}]})$ with for $i \in \mu(g)$, $D_i\langle\vec{u_i}\rangle$ the unique decomposition of s_i, and for $i \notin \mu(g)$, $D_i = \Box$ and $\vec{u_i} = s_i$. Hence $g(\vec{s})^{\circledbullet} = g(\overrightarrow{D[\vec{u}^{\circledbullet}]})^\bullet$ per construction of the decomposition and by definition of $\circledbullet$. To conclude to $g(\vec{s^{\circledbullet}})^\bullet = g(\vec{s})^{\circledbullet} = g(\overrightarrow{D[\vec{u}^{\circledbullet}]})^\bullet$ it then suffices to show that $g(\vec{s^{\circledbullet}})$ and $g(\overrightarrow{D[\vec{u}^{\circledbullet}]})$ are $\hookrightarrow$-convertible since $\hookrightarrow$ is complete by assumption **(ii)**. Convertibility follows from that for each active argument $i \in \mu(g)$ we have that s_i uniquely decomposes as $D_i\langle\vec{u_i}\rangle$ so that $s_i^{\circledbullet} = D_i\langle\vec{u_i}^{\circledbullet}\rangle^\bullet$ hence $s_i^{\circledbullet}$ and $D_i\langle\vec{u_i}^{\circledbullet}\rangle$ are $\hookrightarrow$-convertible and by i being active this extends to the respective ith arguments of $g(\vec{s^{\circledbullet}})$ and $g(\overrightarrow{D[\vec{u}^{\circledbullet}]})$, and from that for each frozen argument $i \notin \mu(g)$ we have by definition of D_i and $\vec{u_i}$ that $s_i^{\circledbullet} = D_i[\vec{u_i}^{\circledbullet}]$. □

Theorem 10. *$\to$ has the Z-property for $\circledbullet$.*

Proof. We have to show that if $\phi : t \to s$ is a TRS step, then there are reductions $s \twoheadrightarrow t^{\circledbullet}$ and $t^{\circledbullet} \twoheadrightarrow s^{\circledbullet}$, giving rise to the Z in [6, Figures 1 and 5]. This we prove by induction on the decomposition $C\langle\vec{t}\rangle$ of the source t of ϕ and by cases on whether or not ϕ is a μ-step.

- if $t \hookrightarrow s$, then by definition of $\circledbullet$ and extensivity of $\circledbullet$, there is a reduction $t \twoheadrightarrow t^{\circledbullet}$ that decomposes into a reduction $\gamma : C\langle\vec{t}\rangle \twoheadrightarrow C\langle\vec{t^{\circledbullet}}\rangle$ with steps at depth at least 1, followed by a reduction $\delta : C\langle\vec{t^{\circledbullet}}\rangle \hookrightarrow\!\!\!\!\rightarrow C\langle\vec{t^{\circledbullet}}\rangle^\bullet = t^{\circledbullet}$ with steps at depth 0. Since ϕ is a step at depth 0, assumption **(i)** yields it and its residuals (after any prefix of γ) are orthogonal to (the corresponding suffix of) γ, giving rise by standard residual theory [8, Chapter 8] to a valley completing the peak between ϕ and γ that comprises a step $\phi/\gamma : C\langle\vec{t^{\circledbullet}}\rangle \hookrightarrow u$ and reduction $\gamma/\phi : s \twoheadrightarrow u$ for some term u.

  To conclude to $s \twoheadrightarrow t^{\circledbullet}$ we compose $\gamma/\phi : s \twoheadrightarrow u$ with the $\hookrightarrow$-reduction (lifted to a $\to$-reduction using $\hookrightarrow \subseteq \to$) of its target u to $\hookrightarrow$-normal form, which is $t^{\circledbullet}$ since $t^{\circledbullet} = C\langle\vec{t^{\circledbullet}}\rangle^\bullet = u^\bullet$ by definition respectively ϕ/γ and completeness of $\hookrightarrow$.

  To conclude to $t^{\circledbullet} \twoheadrightarrow s^{\circledbullet}$, we claim that u has shape $E[\vec{u}^{\circledbullet}]$ and s has shape $E[\vec{u}]$ for some context E and vector of terms $\vec{u}$. Then, composing $\phi/\gamma : C\langle\vec{t^{\circledbullet}}\rangle \hookrightarrow u$ with $u =$

[7] Recall from [6] that a rewrite system $\hookrightarrow$ has the Z-property for a map $\bullet$ on its objects, if $a \hookrightarrow b$ entails $b \hookrightarrow\!\!\!\!\rightarrow a^\bullet \hookrightarrow\!\!\!\!\rightarrow b^\bullet$.

[8] In [4] this is called the *maximal replacing context* and denoted by MRC^μ.

$E[\vec{u}^{\circledbullet}] \twoheadrightarrow E[\vec{u}]^{\circledbullet} = s^{\circledbullet}$ obtained by Lem. 9, yields $C\langle\vec{t}^{\circledbullet}\rangle \twoheadrightarrow s^{\circledbullet}$. From this we conclude to $t^{\circledbullet} = C\langle\vec{t}^{\circledbullet}\rangle^{\bullet} \twoheadrightarrow (s^{\circledbullet})^{\bullet} = s^{\circledbullet}$ by Lem. 7 and idempotence of $\bullet$.

It remains to prove the claim that u has shape $E[\vec{u}^{\circledbullet}]$ and s has shape $E[\vec{u}]$ for some context E and vector of terms $\vec{u}$. The idea is that both C and ℓ are preserved under non-μ-steps, so their *join* is so too, and we set E be the result of contracting ℓ in the join. Formally, we construct E as follows. Let $\varsigma :=$ **let** $X = C[\vec{x}]$ **in** $X(\vec{t})$ be the *cluster* [3] corresponding to the occurrence of the context C in t, and let ζ be the cluster of shape **let** $Y = \ell$ **in** ... corresponding to the occurrence in t of the left-hand side ℓ of the rule $\ell \to r$ contracted in the step $\phi : t \hookrightarrow s$. Their join $\xi := \varsigma \sqcup \zeta$ has shape **let** $Z = D[\vec{z}]$ **in** $Z(\vec{u})$ for some context D and terms $\vec{u}$, by ς being a root cluster of ς having overlap with ζ.

Per construction of ξ and by the TRS $\mathcal{T}$ being left-linear, there is some step ψ from $D[\vec{z}]$ contracting the occurrence of ℓ, such that ϕ is a substitution instance of ψ.[9] Then we define E from the target of ψ writing that uniquely as $E[\vec{w}]$ for $\vec{w}$ comprising the replicated variables of $\vec{z}$, so that $\psi : D[\vec{z}] \hookrightarrow E[\vec{w}]$. In turn, we define $\vec{u}$ from the target s of $\phi : t \hookrightarrow s$, noting the latter can be written as the unique substitution instance $E[\vec{w}]^{\upsilon} = E[\vec{u}]$ of the target $E[\vec{w}]$ of ψ, for substitution υ mapping z_i to u_i such that $\phi = \psi^{\upsilon}$. Per construction, $t = D[\vec{z}]^{\upsilon}$ and $s = E[\vec{w}]^{\upsilon} = E[\vec{u}]$.

Finally, we must show that $u = E[\vec{u}^{\circledbullet}]$. To that end, note that any $\hookrightarrow$-step ϕ' of shape ψ^{σ} for term substitution σ, is orthogonal to any non-μ-step χ having the same source, as (the redex-pattern of) χ can neither have overlap with ς by χ being non-μ, nor have overlap with ζ by assumption **(i)** using that ψ is at depth 0 and χ at depth at least 1, so χ cannot have overlap with their join $\varsigma \sqcup \zeta$ either. Thus, χ is of shape $D[\vec{z}]^{\tau}$ for some step-substitution[10] τ, and $\chi/\phi' = E[\vec{w}]^{\tau}$ and $\phi'/\chi = \psi^{\tau'}$ with τ' the step-substitution such that $\tau'(z_i)$ is the target of $\tau(z_i)$, for all i.

By induction on the length of γ, we obtain from the above that the reduction $\gamma : t = C\langle\vec{t}\rangle \twoheadrightarrow C\langle\vec{t}^{\circledbullet}\rangle$, comprises only steps that are substitution instances of $D[\vec{z}]$ so that $C\langle\vec{t}^{\circledbullet}\rangle$ is as well. In particular note that each reduction from t_i to $t_i^{\bullet}$ does not change its top part (if any) overlapping the occurrence of ℓ, so is the same as that top part where all its arguments have been reduced to $\circledbullet$-normal form. That is, $C\langle\vec{t}^{\circledbullet}\rangle$ has shape $D[\vec{z}]^{\upsilon^{\circledbullet}}$. By the above, u then has shape $E[\vec{w}]^{\upsilon^{\circledbullet}} = E[\vec{u}^{\circledbullet}]$ as common target of ϕ/γ and γ/ϕ, as claimed.

- if $t \to s$ is not a μ-step then $s = C\langle\vec{s}\rangle$ with $t_i \to s_i$ for some i and $t_j = s_j$ for all $j \neq i$. Then the Z-property holds for $\vec{s}$, i.e. $\vec{s} \twoheadrightarrow \vec{t}^{\circledbullet} \twoheadrightarrow \vec{s}^{\circledbullet}$ since by the IH $s_i \twoheadrightarrow t_i^{\circledbullet} \twoheadrightarrow s_i^{\circledbullet}$, and $s_j \twoheadrightarrow t_j^{\circledbullet} = s_j^{\circledbullet}$ for all $j \neq i$ by extensivity of $\circledbullet$. We conclude to $s = C\langle\vec{s}\rangle \twoheadrightarrow C\langle\vec{t}^{\circledbullet}\rangle \twoheadrightarrow C\langle\vec{t}^{\circledbullet}\rangle^{\bullet} = t^{\circledbullet} \twoheadrightarrow C\langle\vec{s}^{\circledbullet}\rangle^{\bullet} = s^{\circledbullet}$, using that the Z-property holds for $\vec{s}$ by the IH and closure of $\to$ under contexts for the first reduction, extensivity of $\bullet$ and $\hookrightarrow \subseteq \to$ for the second, and Z for $\vec{s}$ and closure under contexts and preservation of $\twoheadrightarrow$ by $\bullet$ for the third. □

From this we immediately obtain by [6, Lem. 51 and Thm. 50] that:

[9] Using traditional unification-speak D can be described as being obtained by unifying the occurrence of the left-hand side ℓ with the context C (both linear and renamed apart). E is then the result of contracting the ℓ-redex in D. We prefer to employ the lattice-theoretic language developed in [3] as that is based on encompassment which encompasses both the subsumption (prefix; unification) and the superterm (suffix) orders employed in such traditional accounts, and moreover avoids context-talk which is imprecise here since D and E are not simply contexts, but linear terms; in particular, the names of the holes in E do matter.

[10] A substitution τ such that for all i, $\tau(z_i)$ either is a single step or a term.

Corollary 11. *Under assumptions (i) and (ii), $\to$ is confluent and the bullet strategy $\multimap\!\!\!\bullet\!\!\to$, is a hyper-cofinal strategy.*[11]

Thus the bullet strategy $\multimap\!\!\!\bullet\!\!\to$ is (hyper-)normalising [8]. Moreover, the layered bullet function $\circledbullet$ induces an *effective* (if $\hookrightarrow$ is) confluence construction and cofinal strategy.

A concrete criterion Our approach to confluence of a term rewrite system (via the Z-property) has confluence of context-sensitive rewriting $\hookrightarrow$ as an assumption; in fact local confluence suffices given termination is also assumed.[12] The following is a sufficient condition for local confluence of context-sensitive rewriting $\hookrightarrow$ known from [5]; see that paper for others.

(iii) $\mathcal{T},\mu$ is 0-*preserving* if, whenever a variable occurs at depth 0 in the left-hand side of a rule, then all its occurrences in the right-hand side are at depth 0 as well.

Lemma 12. *If $\mathcal{T},\mu$ is a left-linear CSR satisfying assumptions (i) and (iii) with $\hookrightarrow$-joinable critical peaks, then context-sensitive rewriting $\hookrightarrow$ is locally confluent.*

Proof. This is the special case of [5, Thm. 30], arising by (additionally) assuming left-linearity and the absence of extended critical pairs [5, Definition 29].

For a direct proof, note that a local $\hookrightarrow$-peak $s \hookleftarrow t \hookrightarrow u$ either is overlapping or not.

In the former case, the peak is an instance of a critical $\hookrightarrow$-peak occurring in some context at an active position. Then we conclude by assumption **(i)** and $\hookrightarrow$-joinability of critical peaks.

The latter case further splits into the disjoint (a) and nested redex-patterns cases (b) and (b') in the proof of [8, Lem. 2.7.15], Huet's Critical Pair Lemma. The proof of case (a) carries over directly from $\to$ to $\hookrightarrow$. The proof of cases (b) and (b') carries over as well, but using assumption **(iii)** to ensure that the residuals (at parallel positions) of the nested step remain at depth 0, so are $\hookrightarrow$-steps again. □

Since convectivity entails assumption **(i)**, and $\hookrightarrow$-joinability of critical peaks and 0-preservingness entail confluence of $\hookrightarrow$ for left-linear CSRs by Lem. 12, combining this with termination of $\mathcal{T}$ all assumptions of Thm. 10 are satisfied:

Corollary 13. *If $\mathcal{T},\mu$ is a left-linear 0-preserving CSR such that μ is convective, critical peaks are $\hookrightarrow$-joinable, and context-sensitive rewriting $\hookrightarrow$ is terminating, then the TRS $\mathcal{T}$, i.e. the rewrite system $\to$, has the Z-property for the layered bullet function $\circledbullet$.*

This generalises [1, Thm. 2], the main result of that paper, both by *relaxing* two of its assumptions, canonicity to convectivity and level-decreasingness to 0-preservingness, and by *strengthening* its conclusion from confluence to the Z-property.

Example 14 (application to running example)**.** *The CSR of Ex. 4 is left-linear (by inspection of the left-hand sides; no repeated variables), 0-preserving (vacuously so, since there are no variables at depth 0 in left-hand sides; all occur in overlined subterms), has a convective replacement map (μ^{cnv} is the most restrictive such), and is terminating as was observed.*

The (only) critical peak is between its fifth and sixth rules and is $\hookrightarrow$-joinable as shown by (the following) two legs of its confluence diagram: $\mathsf{inc}(\mathsf{tl}(\mathsf{from}(\overline{x}))) \hookrightarrow \mathsf{tl}(\mathsf{inc}(\mathsf{from}(\overline{x}))) \hookrightarrow \mathsf{tl}(\mathsf{inc}(\overline{x} : \overline{\mathsf{from}(\mathsf{s}(\overline{x}))}))) \hookrightarrow \mathsf{tl}(\overline{\mathsf{s}(\overline{x})} : \overline{\mathsf{inc}(\mathsf{from}(\overline{\mathsf{s}(\overline{x})}))})) \hookrightarrow \mathsf{inc}(\mathsf{from}(\overline{\mathsf{s}(\overline{x})}))$ *and* $\mathsf{inc}(\mathsf{tl}(\mathsf{from}(\overline{x}))) \hookrightarrow \mathsf{inc}(\mathsf{tl}(\overline{x} :$

[11] Recall from [6] that the *bullet* strategy iterates the bullet map (here $\circledbullet$) on objects, and that a $\to$-strategy is *hyper-cofinal* if for any $a \twoheadrightarrow b$, starting from a always eventually performing a $\multimap\!\!\!\bullet\!\!\to$-step after a number of $\to$-steps will yield an object c that *exceeds* b in the sense that $b \twoheadrightarrow c$.

[12] Alternatively, completeness may be decomposed into *random descent* and *normalisation* [7].

$\overline{\mathsf{from}(\overline{\mathsf{s}(\overline{x})})}) \hookrightarrow \mathsf{inc}(\mathsf{from}(\overline{\mathsf{s}(\overline{x})}))$.[13] *Corollary 13 yields $\to$ has the Z-property, is confluent, and $\multimap\!\!\!\!\bullet\!\!\to$ is a cofinal $\twoheadrightarrow$-strategy.*

Remark 15. The methods of [1] do not apply to yield the result of Ex. 14. Their methods require level-decreasingness of the rules and the fifth added rule is not for the canonical replacement map μ^{can} employed by them: the level of x in the lhs is then 1 whereas in the rhs it occurs not only with level 1 but also with level 3. The only way to regain level-decreasingness is to make both the second argument of : and the argument of s active, but that would violate termination of $\hookrightarrow$ (the fifth rule becomes spiralling), one of the other assumptions of [1, Thm. 2].

Conclusion By relaxing the assumptions of [1, Thm. 2], Cor. 13 *partially* settles [1, Open Problem 1]. In the long draft `http://www.javakade.nl/research/pdf/z-csr.pdf` from which this note was derived, we provided several more illustrative examples (omitted here for compactness) and also positively settled [1, Open Problem 2]. Although we didn't implement our method, we think it should be relatively easy to integrate into extant tools for context-sensitive rewriting (especially those already basing themselves on canonical replacement maps).

Acknowledgments We thank Nao Hirokawa and Salvador Lucas for inspiration and feedback.

References

[1] B. Gramlich and S. Lucas. Generalizing Newman's lemma for left-linear rewrite systems. In F. Pfenning, editor, *Term Rewriting and Applications, 17th International Conference, RTA 2006, Seattle, WA, USA, August 12-14, 2006, Proceedings*, volume 4098 of *LNCS*, pages 66–80. Springer, 2006.

[2] N. Hirokawa. Seven confluence criteria for solving COPS #20. In C. Rocha and S. Winkler, editors, *11th International Workshop on Confluence, IWC 2022, August 1, 2022, Haifa, Israel*, page 1, 2022. Invited talk.

[3] N. Hirokawa, J. Nagele, V. van Oostrom, and M. Oyamaguchi. Confluence by critical pair analysis revisited. In *CADE 27*, volume 11716 of *LNCS*, pages 319–336. Springer, 2019.

[4] S. Lucas. Derivational complexity and context-sensitive rewriting. *Journal of Automated Reasoning*, 65(8):1191–1229, 2021.

[5] S. Lucas, M. Vítores, and R. Gutiérrez. Proving and disproving confluence of context-sensitive rewriting. *Journal of Logical and Algebraic Methods in Programming*, 126:100749, 2022.

[6] V. van Oostrom. Z; syntax-free developments. In N. Kobayashi, editor, *6th International Conference on Formal Structures for Computation and Deduction, FSCD 2021, July 17-24, 2021, Buenos Aires, Argentina (Virtual Conference)*, volume 195 of *LIPIcs*, pages 24:1–24:22. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021.

[7] V. van Oostrom. Uniform completeness. In *IWC 2022*, pages 19–24, 2022.

[8] Terese. *Term Rewriting Systems*. Cambridge University Press, 2003.

A CSR of Ex. 14 in format suitable for automation

The CSR of Ex. 14 can be given in *COPS*[14] format as:

[13] Indeed, as pointed out by Salvador Lucas (personal communication 1-6-2023) termination and local confluence of this CSR (see App. A) are established automatically by the tool CONFident (`http://zenon.dsic.upv.es/confident/`), employing the general criterion mentioned in the proof of Lem. 12.

[14] See `http://project-coco.uibk.ac.at/problems/`.

```
(REPLACEMENT-MAP
  (_:_ )
  (from )
  (hd )
  (s )
)
(VAR x y)
(RULES
nats -> from(0)
inc(_:_(x,y)) -> _:_(s(x),inc(y))
hd(_:_(x,y)) -> x
tl(_:_(x,y)) -> y
from(x) -> _:_(x,from(s(x)))
inc(tl(from(x))) -> tl(inc(from(x)))
)
```