Confluence by Critical Pair Analysis Revisited

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Term Rewrite System (TRS)

TRS

$$\begin{aligned} 1: \quad & \mathsf{0} \cdot x \to \mathsf{0} \\ 2: \quad & (x \cdot y) \cdot z \to x \cdot (y \cdot z) \end{aligned}$$

rewriting

$(\mathsf{0} \cdot x) \cdot (\mathsf{0} \cdot y)$

Term Rewrite System (TRS)

TRS

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\begin{aligned}
1: &\quad 0 \cdot x \to 0 \\
2: &\quad (x \cdot y) \cdot z \to x \cdot (y \cdot z)
\end{aligned}
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rewriting

$$
\begin{array}{c}
(0 \cdot x) \cdot (0 \cdot y) \\
{\scriptstyle 2}\big\downarrow \\
0 \cdot (x \cdot (0 \cdot y))
\end{array}
$$

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(0 \cdot x) \cdot (0 \cdot y) \\
{\scriptstyle 2}\big\downarrow \\
0 \cdot (x \cdot (0 \cdot y)) \\
{\scriptstyle 1}\big\downarrow \\
0 \cdot (x \cdot 0)
\end{array}
$$

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(0 \cdot x) \cdot (0 \cdot y) \\
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0 \cdot (x \cdot (0 \cdot y)) \\
{\scriptstyle 1}\big\downarrow \\
0 \cdot (x \cdot 0) \\
{\scriptstyle 1}\big\downarrow \\
0
\end{array}
$$

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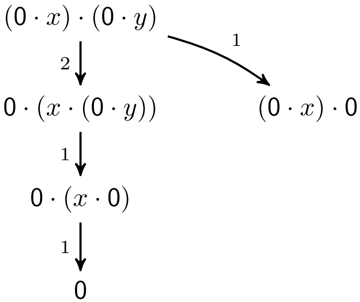

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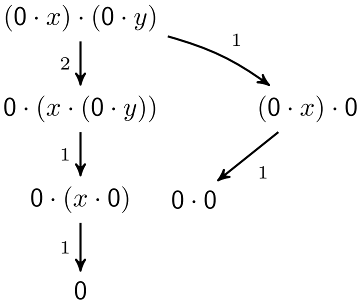

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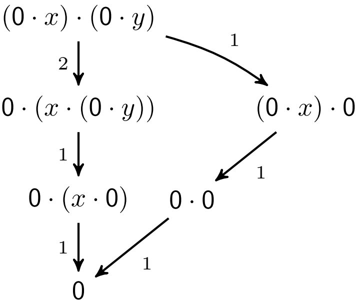

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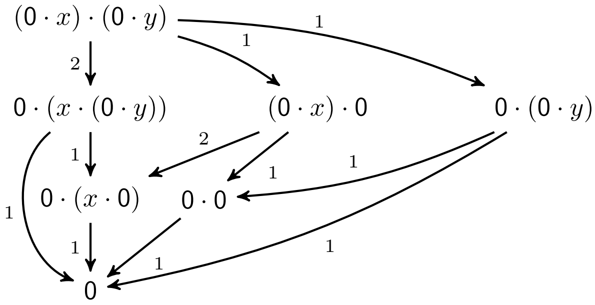

Confluence

Definition

confluence

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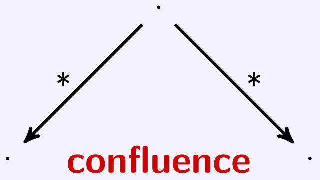

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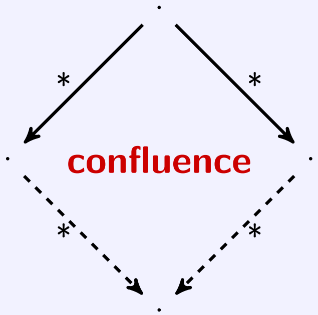

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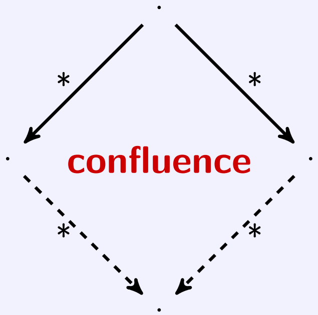
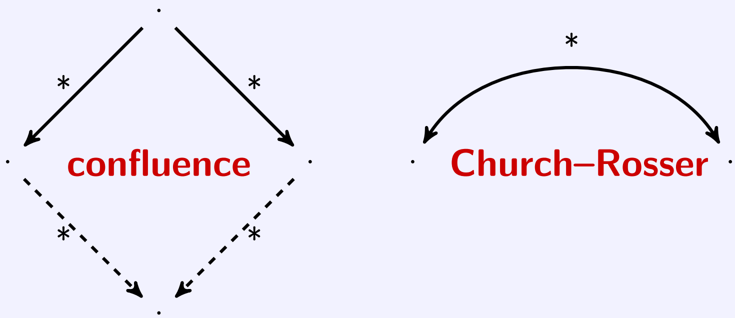


Church–Rosser

Confluence

Definition



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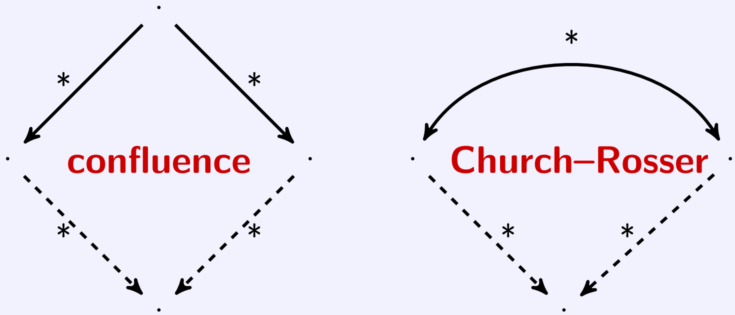

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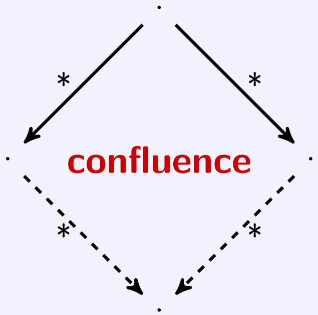

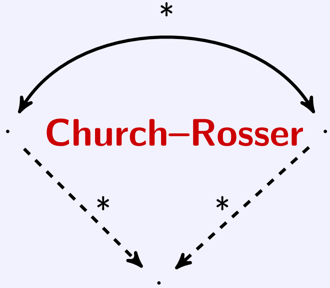

Fact

confluence and Church–Rosser property are equivalent

Knuth–Bendix' Criterion (1970)

Theorem

terminating *TRS is confluent if all critical pairs are joinable*

Knuth–Bendix' Criterion (1970)

Theorem

terminating *TRS is confluent if all critical pairs are joinable*

Proof.

$\leftarrow \cdot \rightarrow$ is decreasing with source labelling wrt $\rightarrow^+$ □

we show confluence of TRS $\mathcal{R}$

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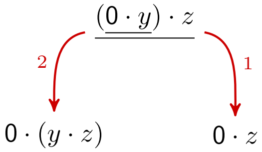

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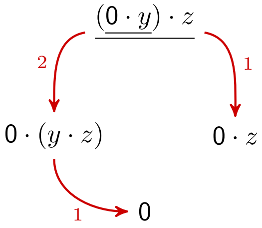

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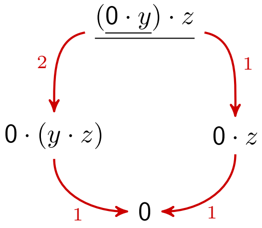

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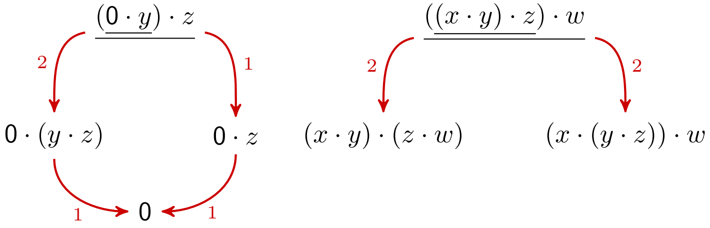

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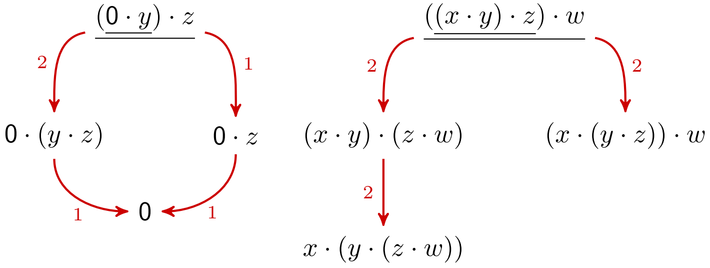

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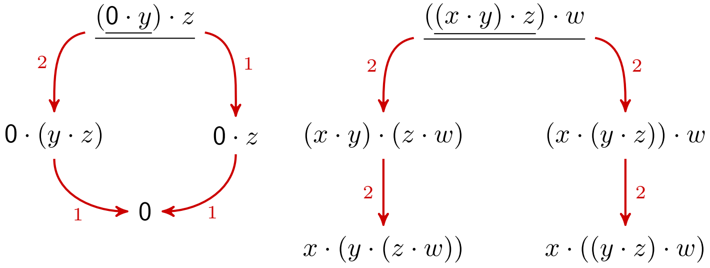

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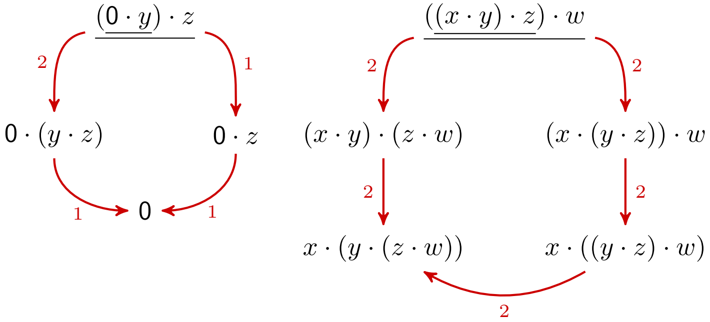

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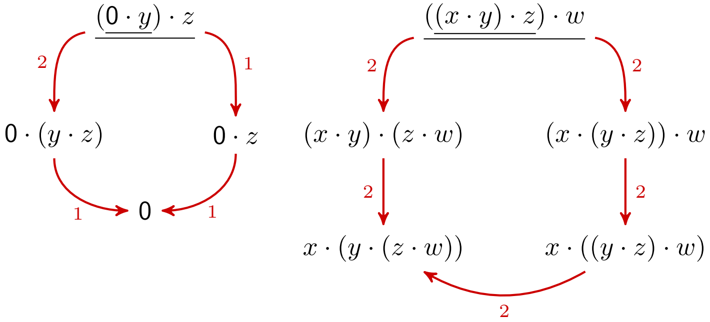

3. hence $\mathcal{R}$ is confluent

Motivating Example

TRS

$$
\begin{array}{rrlrrl}
1: & \mathsf{nats} & \to \mathsf{0} : \mathsf{inc}(\mathsf{nats}) & 4: & \mathsf{inc}(\mathsf{tl}(\mathsf{nats})) & \to \mathsf{tl}(\mathsf{inc}(\mathsf{nats})) \\
2: & \mathsf{d}(x) & \to x : (x : \mathsf{d}(x)) & 5: & \mathsf{hd}(x : y) & \to x \\
3: & \mathsf{inc}(x : y) & \to \mathsf{s}(x) : \mathsf{inc}(y) & 6: & \mathsf{tl}(x : y) & \to y
\end{array}
$$

Motivating Example

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\begin{array}{rlrl}
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\end{array}
$$

rewriting

$\mathsf{inc}(\mathsf{0} : \mathsf{s}(\mathsf{0}) : \mathsf{s}(\mathsf{s}(\mathsf{0})) : \mathsf{nil})$

Motivating Example

TRS

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\begin{array}{rlrl}
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\end{array}
$$

rewriting

$\mathsf{inc}(\mathsf{0} : \mathsf{s}(\mathsf{0}) : \mathsf{s}(\mathsf{s}(\mathsf{0})) : \mathsf{nil})$

Motivating Example

TRS

$1:$	$\mathsf{nats} \to \mathsf{0} : \mathsf{inc}(\mathsf{nats})$	$4:$	$\mathsf{inc}(\mathsf{tl}(\mathsf{nats})) \to \mathsf{tl}(\mathsf{inc}(\mathsf{nats}))$
$2:$	$\mathsf{d}(x) \to x : (x : \mathsf{d}(x))$	$5:$	$\mathsf{hd}(x : y) \to x$
$3:$	$\mathsf{inc}(x : y) \to \mathsf{s}(x) : \mathsf{inc}(y)$	$6:$	$\mathsf{tl}(x : y) \to y$

rewriting

$\mathsf{inc}(\mathsf{0} : \mathsf{1} : \mathsf{2} : \mathsf{nil})$

Motivating Example

TRS

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\begin{array}{llll}
1: & \mathsf{nats} \to \mathsf{0} : \mathsf{inc}(\mathsf{nats}) & 4: & \mathsf{inc}(\mathsf{tl}(\mathsf{nats})) \to \mathsf{tl}(\mathsf{inc}(\mathsf{nats})) \\
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rewriting

$$\mathsf{inc}(\mathsf{0} : \mathsf{1} : \mathsf{2} : \mathsf{nil}) \xrightarrow{*} \mathsf{1} : \mathsf{2} : \mathsf{3} : \mathsf{inc}(\mathsf{nil})$$

Motivating Example

TRS

$1:\quad \mathsf{nats} \to \mathsf{0} : \mathsf{inc}(\mathsf{nats})$

$2:\quad \mathsf{d}(x) \to x : (x : \mathsf{d}(x))$

$3:\quad \mathsf{inc}(x : y) \to \mathsf{s}(x) : \mathsf{inc}(y)$

$4:\quad \mathsf{inc}(\mathsf{tl}(\mathsf{nats})) \to \mathsf{tl}(\mathsf{inc}(\mathsf{nats}))$

$5:\quad \mathsf{hd}(x : y) \to x$

$6:\quad \mathsf{tl}(x : y) \to y$

rewriting

$$\mathsf{inc}(\mathsf{0} : \mathsf{1} : \mathsf{2} : \mathsf{nil}) \xrightarrow{*} \mathsf{1} : \mathsf{2} : \mathsf{3} : \mathsf{inc}(\mathsf{nil})$$

nats

Motivating Example

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rewriting

$$\mathsf{inc}(\mathsf{0} : \mathsf{1} : \mathsf{2} : \mathsf{nil}) \xrightarrow{*} \mathsf{1} : \mathsf{2} : \mathsf{3} : \mathsf{inc}(\mathsf{nil})$$

$$\mathsf{nats} \xrightarrow{*} \mathsf{0} : \mathsf{1} : \mathsf{2} : \mathsf{inc}(\mathsf{inc}(\mathsf{inc}(\mathsf{nats})))$$

Motivating Example

TRS

$$\begin{aligned} 1\colon\ & \mathsf{nats} \to \mathsf{0} : \mathsf{inc}(\mathsf{nats}) & 4\colon\ & \mathsf{inc}(\mathsf{tl}(\mathsf{nats})) \to \mathsf{tl}(\mathsf{inc}(\mathsf{nats})) \\ 2\colon\ & \mathsf{d}(x) \to x : (x : \mathsf{d}(x)) & 5\colon\ & \mathsf{hd}(x : y) \to x \\ 3\colon\ & \mathsf{inc}(x : y) \to \mathsf{s}(x) : \mathsf{inc}(y) & 6\colon\ & \mathsf{tl}(x : y) \to y \end{aligned}$$

rewriting

$$\begin{aligned} \mathsf{inc}(\mathsf{0} : \mathsf{1} : \mathsf{2} : \mathsf{nil}) &\xrightarrow{*} \mathsf{1} : \mathsf{2} : \mathsf{3} : \mathsf{inc}(\mathsf{nil}) \\ \mathsf{nats} &\xrightarrow{*} \mathsf{0} : \mathsf{1} : \mathsf{2} : \mathsf{inc}(\mathsf{inc}(\mathsf{inc}(\mathsf{nats}))) \\ \mathsf{d}(\mathsf{nats}) &\xrightarrow{*} \mathsf{0} : \mathsf{0} : \mathsf{1} : \mathsf{1} : \mathsf{2} : \mathsf{2} : \mathsf{d}(\mathsf{inc}(\mathsf{inc}(\mathsf{inc}(\mathsf{nats})))) \end{aligned}$$

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rewriting

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\begin{aligned}
\mathsf{inc}(\mathsf{0} : \mathsf{1} : \mathsf{2} : \mathsf{nil}) &\xrightarrow{*} \mathsf{1} : \mathsf{2} : \mathsf{3} : \mathsf{inc}(\mathsf{nil}) \\
\mathsf{nats} &\xrightarrow{*} \mathsf{0} : \mathsf{1} : \mathsf{2} : \mathsf{inc}(\mathsf{inc}(\mathsf{inc}(\mathsf{nats}))) \\
\mathsf{d}(\mathsf{nats}) &\xrightarrow{*} \mathsf{0} : \mathsf{0} : \mathsf{1} : \mathsf{1} : \mathsf{2} : \mathsf{2} : \mathsf{d}(\mathsf{inc}(\mathsf{inc}(\mathsf{inc}(\mathsf{nats}))))
\end{aligned}
$$

Question

is this TRS confluent? how to prove it?

Knuth–Bendix' criterion fails:

$$
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- all critical pairs are joinable:

$$\mathsf{tl}(\mathsf{inc}(\mathsf{nats})) \xleftarrow{4} \mathsf{inc}(\mathsf{tl}(\mathsf{nats})) \xrightarrow{1} \mathsf{inc}(\mathsf{tl}(\mathsf{0} : \mathsf{inc}(\mathsf{nats})))$$

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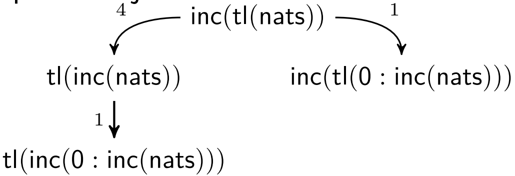

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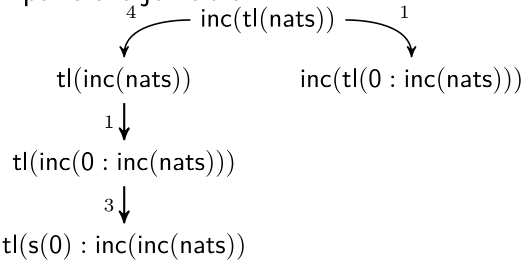

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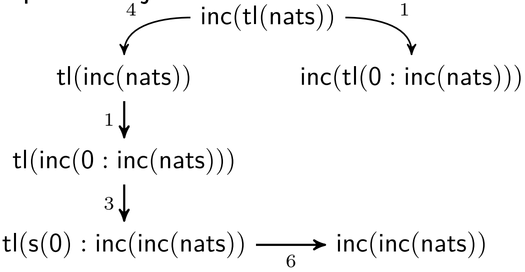

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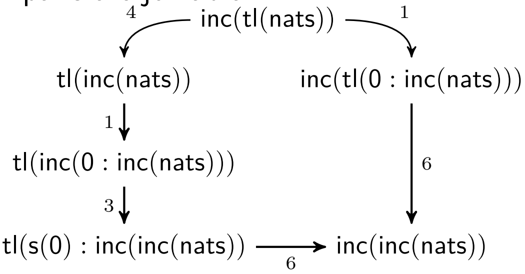

Knuth–Bendix' criterion fails:

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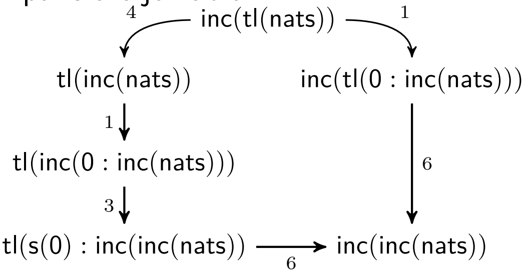

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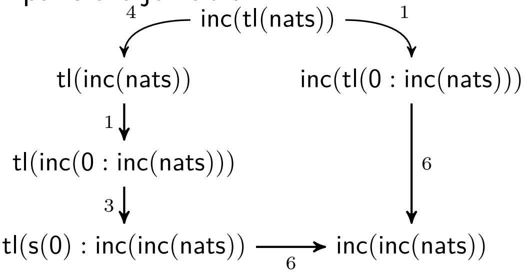

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Question

can't we relax termination requirement?

Obstacle 1: Non-Left-Linear Rules

Huet's non-left-linear TRS:

$$\begin{aligned} 1: &\quad \mathsf{f}(x, x) \to \mathsf{a} \\ 2: &\quad \mathsf{f}(\mathsf{g}(x), x) \to \mathsf{b} \\ 3: &\quad \mathsf{c} \to \mathsf{g}(\mathsf{c}) \end{aligned}$$

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- but TRS is not confluent:

$$\mathsf{a} \xleftarrow{1} \mathsf{f}(\mathsf{c}, \mathsf{c}) \xrightarrow{2} \mathsf{f}(\mathsf{g}(\mathsf{c}), \mathsf{c}) \xrightarrow{3} \mathsf{b}$$

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left-linear but variable-duplicating TRS:

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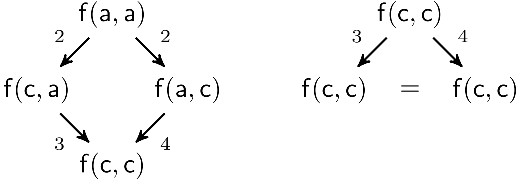

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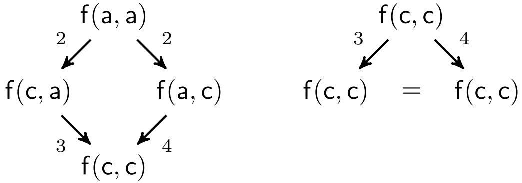

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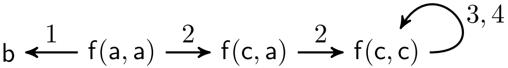

Contributions

Idea

exploit termination of subsystem

Rest of Talk

1. confluence by critical-pair-closing systems
2. confluence by hot-decreasingness
3. new definition of critical overlaps

Confluence by Critical-Pair-Closing Systems

Critical-Pair-Closing Systems

Definition

- TRS $\mathcal{C}$ is **critical-pair-closing** system for $\mathcal{R}$ if
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if $\mathcal{R}$ is left-linear, $\mathcal{C}$ is critical-pair-closing and $\mathcal{C}_\mathsf{d}/\mathcal{R}$ terminates

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$\to_{\mathcal{C}}^* \cdot \multimap_{\mathcal{R}}$ has diamond property □

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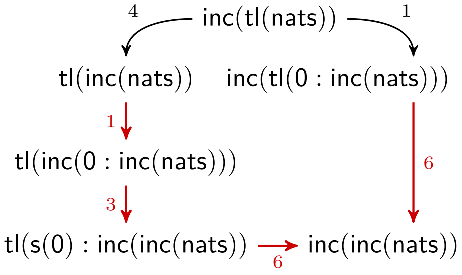

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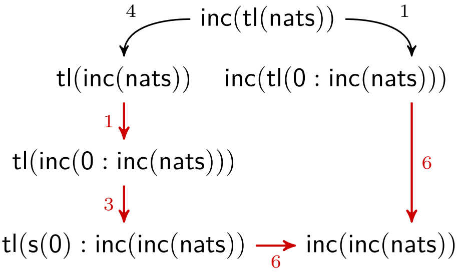

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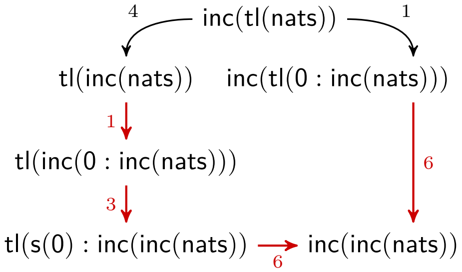

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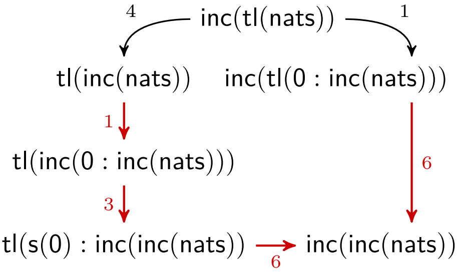

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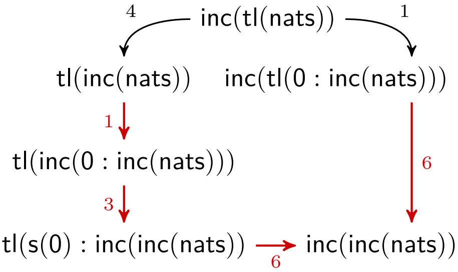

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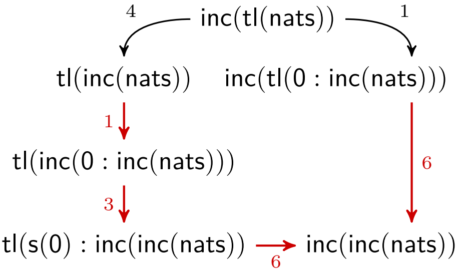

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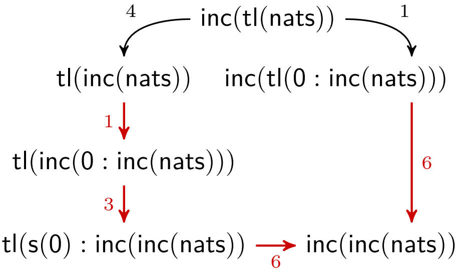

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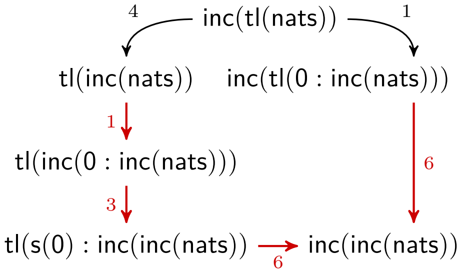

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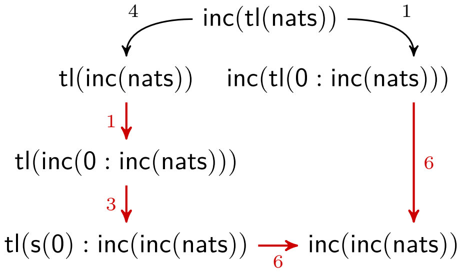

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Generalizing Huet's Strong Closedness

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TRS $\mathcal{R}$ is strongly closed if $\mathsf{CP}(\mathcal{R}) \subseteq \rightarrow_{\mathcal{R}}^{*} \cdot {}_{\mathcal{R}}^{=}\!\leftarrow \cap \rightarrow_{\mathcal{R}}^{=} \cdot {}_{\mathcal{R}}^{*}\!\leftarrow$

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Proof.

$\mathcal{C}_{\mathsf{d}}/\mathcal{R}$ is terminating as $\mathcal{C}_{\mathsf{d}} = \varnothing$ and $\mathcal{C}$ is confluent □

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generalized strong closedness (new)	**94**	15

Confluence by Hot-Decreasingness

generalization of

- Huet's, van Oostrom's, and Toyama's criteria
- Hirokawa and Oyamaguchi's Termination-based criteria

Definition

let $\mathcal{C}$ be terminating subset of TRS $\mathcal{R}$
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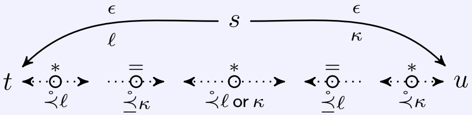

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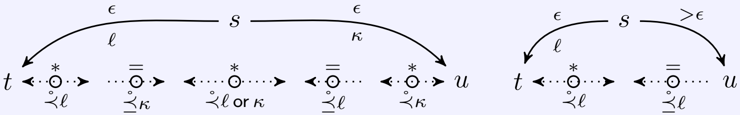

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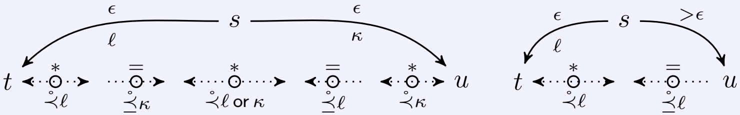

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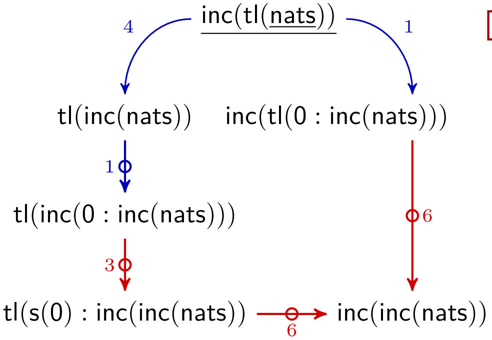

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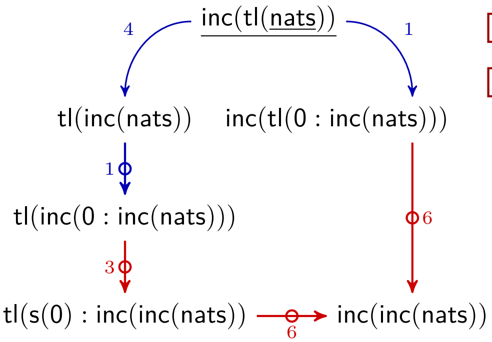

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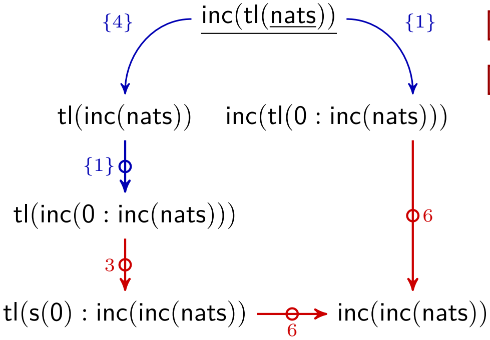

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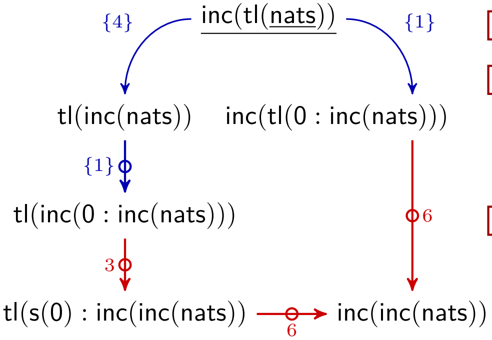

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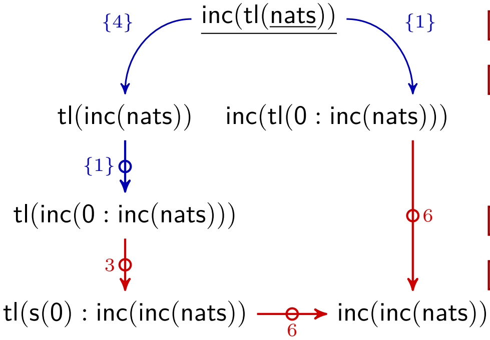

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Corollary (Knuth and Bendix 1970)

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Experimental Results

- 432 left-linear TRSs from COPS
- 224 are known to be confluent and 173 are non-confluent
- employed termination tool NaTT and Yices 2

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Critical Overlaps Revisited

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Our Solution

lattice-theoretic characterization of critical overlap

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for instance,

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$$= \mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{f}(\mathsf{a}))))$$

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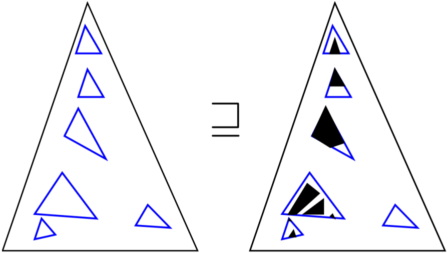

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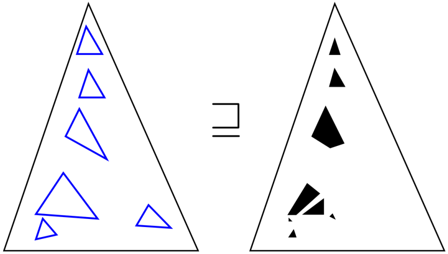

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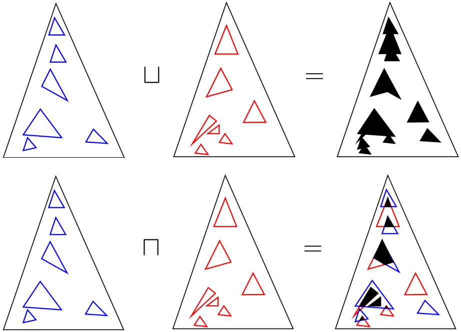

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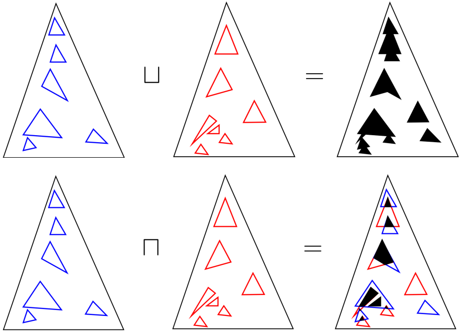

Proposition

linear patterns ς and ζ are critically overlapping if $\varsigma \sqcup \zeta = \top$

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