feebly not weakly

Vincent van Oostrom

HOR, 11.15–11.45, Saturday July 12, 2014

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orthogonalisable

feebly orthogonal

orthogonalisable ⇔ feebly orthogonal

higher-order

orthogonal

rewriting: independence of steps

orthogonal

term rewriting: rules left-linear and no critical peaks

orthogonal

term rewriting: rules left-linear and no critical peaks

Examples

- lambda-calculus with β-reduction

$$@(\lambda(x.F(x)),G) \quad \rightarrow \quad F(G)$$

orthogonal

term rewriting: rules left-linear and no critical peaks

Examples

- lambda-calculus with β-reduction

$$@(\lambda(x.F(x)),G) \rightarrow F(G)$$

- unary natural numbers with rules for predecessor

$$P(0) \rightarrow 0$$
$$P(S(x)) \rightarrow x$$

orthogonal

term rewriting: rules left-linear and no critical peaks

Examples

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  $$@(\lambda(x.F(x)), G) \rightarrow F(G)$$

- unary natural numbers with rules for predecessor

  $$P(0) \rightarrow 0$$
  $$P(S(x)) \rightarrow x$$

- omit parentheses for readability (string rewriting)

  $$P\,0 \rightarrow 0$$
  $$P\,S \rightarrow \varepsilon$$

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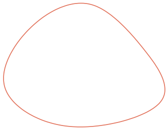
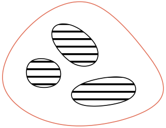
orthogonal ⇒ space of multi-steps



independence via closure under union

orthogonal ⇒ space of multi-steps



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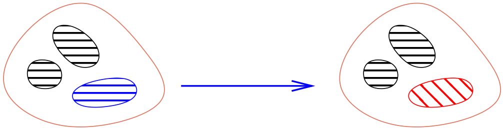

independence via closure under union

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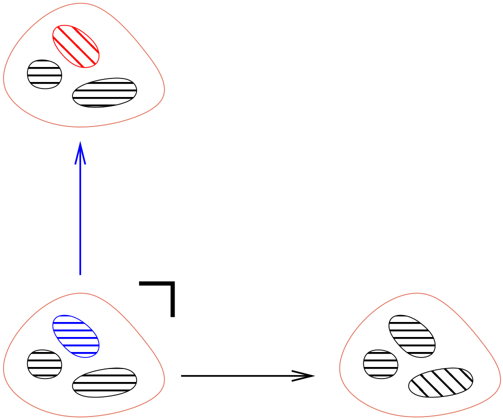

independence via closure under union

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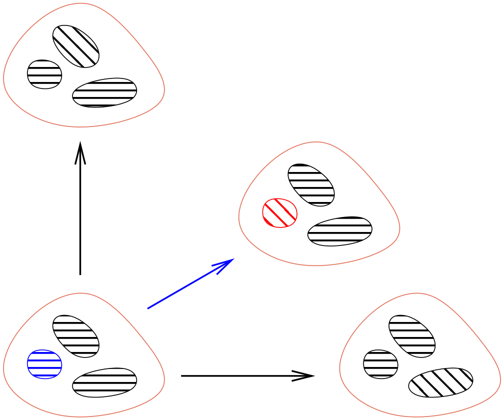
orthogonal ⇒ space of multi-steps



independence via closure under union

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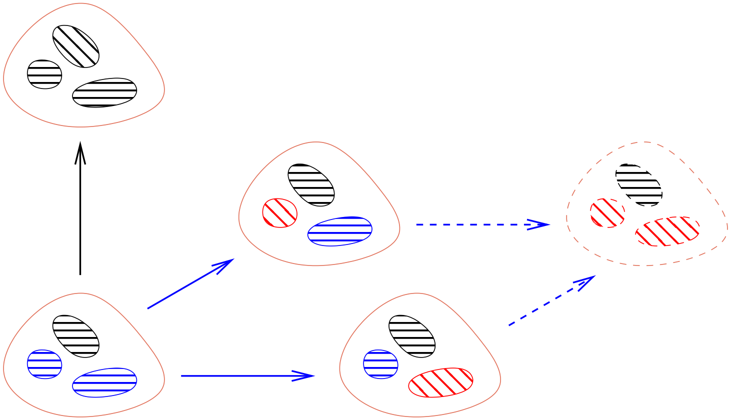

independence via closure under union

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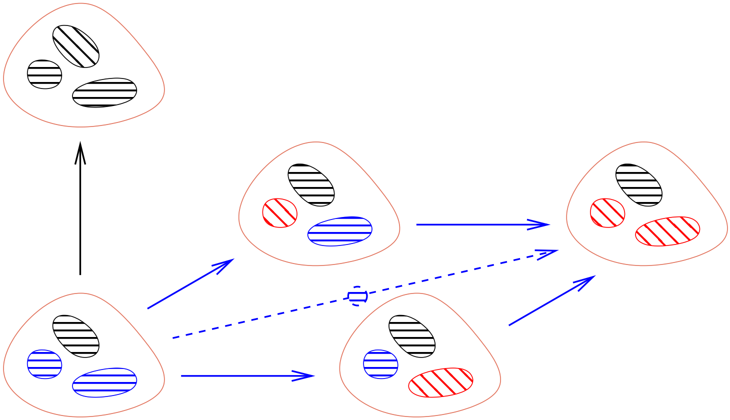
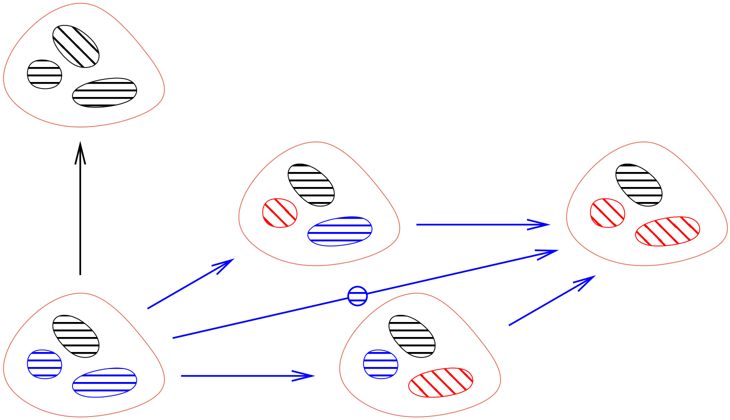


independence via closure under union

orthogonal ⇒ space of multi-steps



independence via closure under union

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orthogonal ⇒ space of multi-steps

$P\,S\,P\,P\,S\,S\,P\,S$

independence via closure under union

orthogonal ⇒ space of multi-steps

$\underline{P\,S}\,P\,\underline{P\,S}\,S\,\underline{P\,S}$

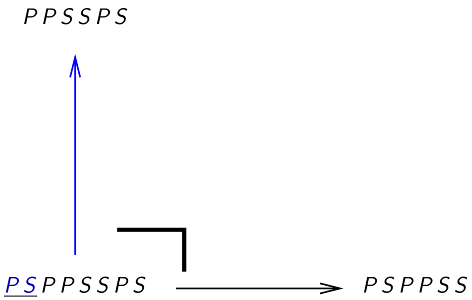
independence via closure under union

orthogonal ⇒ space of multi-steps

$P\,S\,P\,P\,S\,S\,\underline{P\,S} \longrightarrow P\,S\,P\,P\,S\,S$

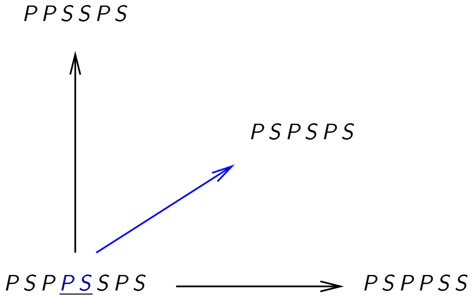
independence via closure under union

orthogonal ⇒ space of multi-steps



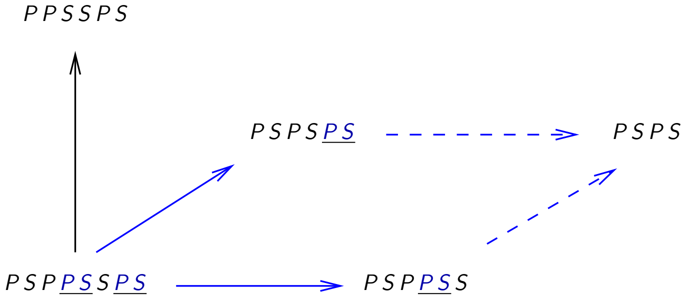
independence via closure under union

orthogonal ⇒ space of multi-steps



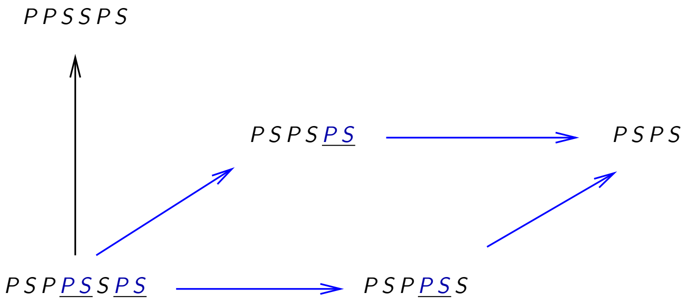
independence via closure under union

orthogonal ⇒ space of multi-steps



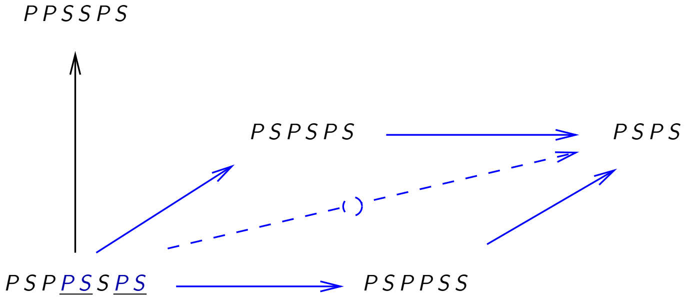
independence via closure under union

orthogonal ⇒ space of multi-steps



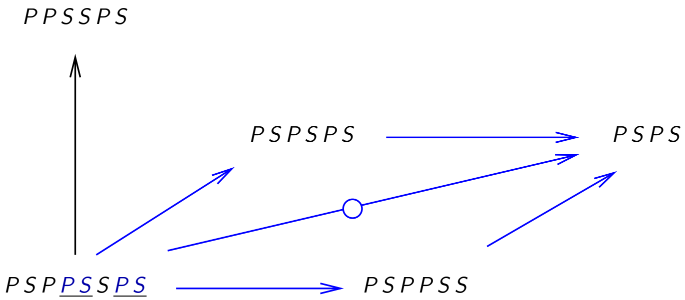
independence via closure under union

orthogonal ⇒ space of multi-steps



independence via closure under union; multi-step to $P\,S\,P\,S$

orthogonal ⇒ space of multi-steps



independence via closure under union

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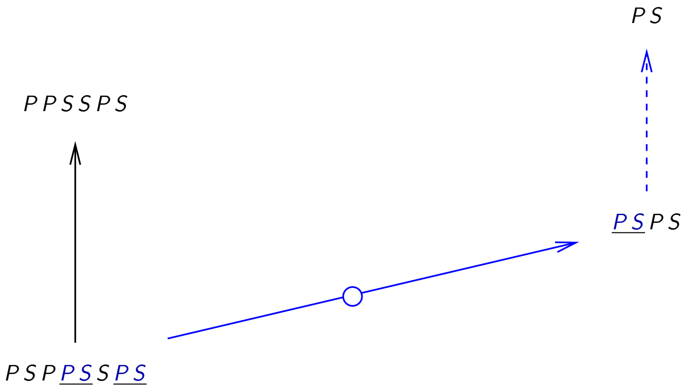
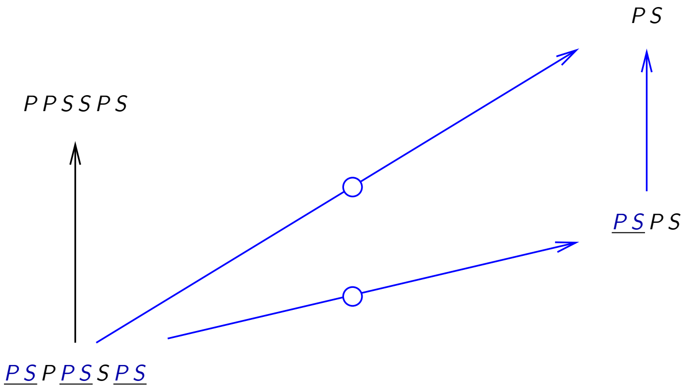
orthogonal ⇒ space of multi-steps



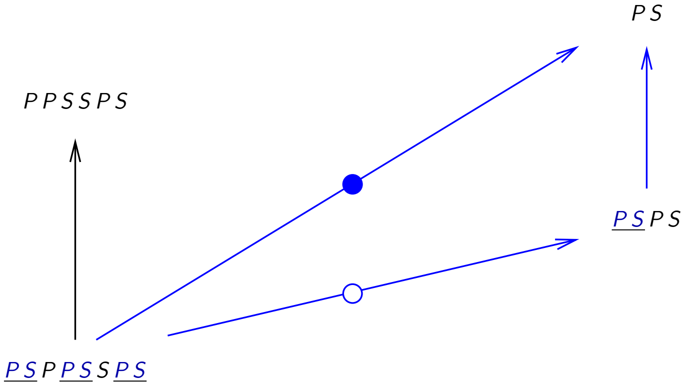
independence via closure under union

orthogonal ⇒ space of multi-steps



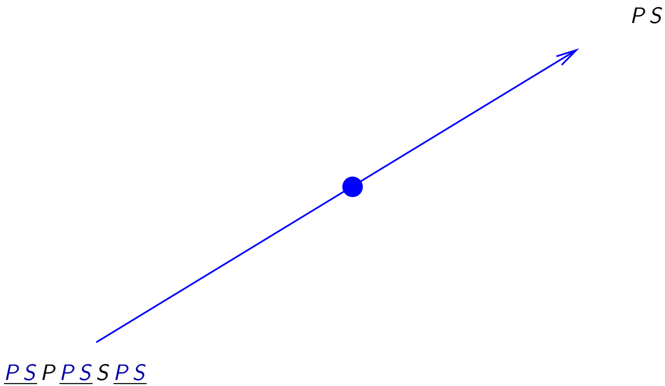
independence via closure under union; $\to \subseteq \multimap\!\!\to \subseteq \twoheadrightarrow$

orthogonal ⇒ space of multi-steps



independence via closure under union

orthogonal ⇒ space of multi-steps



$P\,S\,P\,P\,S\,S\,P\,S \;\multimap\; P\,S$ full multi-step from $P\,S\,P\,P\,S\,S\,P\,S$

space of multi-steps ⇒ confluence, cofinality

a

$\multimap\!\!\!\rightarrow$ has angle property: $\forall a$

space of multi-steps ⇒ confluence, cofinality

$a^\bullet$

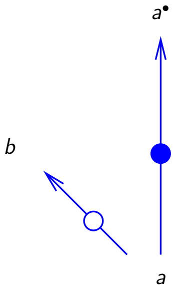
a

$\multimap\!\!\!\to$ has angle property: $\forall a,\ \exists a^\bullet$

space of multi-steps ⇒ confluence, cofinality

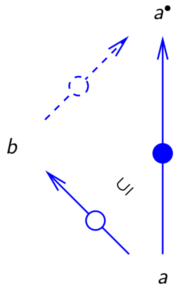
$\multimap\!\!\!\to$ has angle property: $\forall a, \exists a^\bullet$

space of multi-steps ⇒ confluence, cofinality



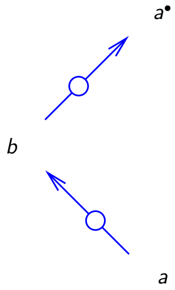
$\multimap\!\!\!\to$ has angle property: $\forall a, \exists a^\bullet, \forall b, a \multimap\!\!\!\to b$

space of multi-steps ⇒ confluence, cofinality



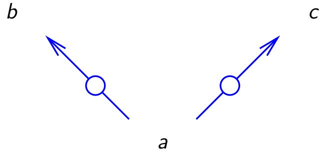
$\mathrel{-\!\!\circ\!\!\to}$ has angle property: $\forall a,\ \exists a^\bullet,\ \forall b,\ a \mathrel{-\!\!\circ\!\!\to} b \Rightarrow b \mathrel{-\!\!\circ\!\!\to} a^\bullet$

space of multi-steps $\Rightarrow$ confluence, cofinality



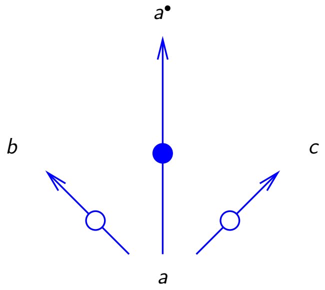
$\mathrel{-\!\!\circ\!\!\to}$ has angle property: $\forall a,\ \exists a^\bullet,\ \forall b,\ a \mathrel{-\!\!\circ\!\!\to} b \Rightarrow b \mathrel{-\!\!\circ\!\!\to} a^\bullet$

space of multi-steps ⇒ confluence, cofinality



angle property ⇒ $\multimap\!\!\to$ has diamond property

space of multi-steps ⇒ confluence, cofinality



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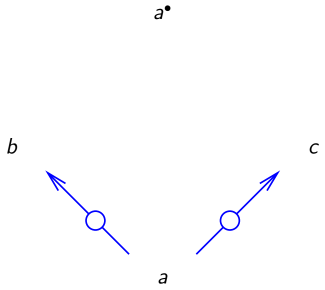

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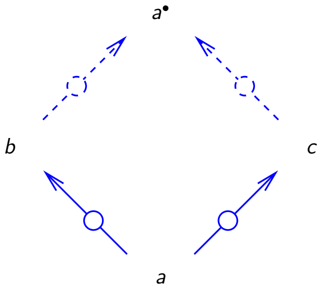

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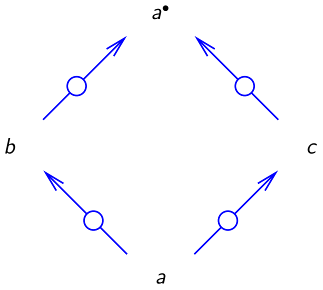
space of multi-steps ⇒ confluence, cofinality



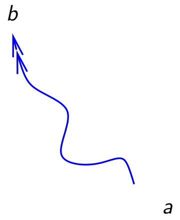
angle property ⇒ confluence of →

space of multi-steps ⇒ confluence, cofinality

a

angle property ⇒ $\forall b,\ a \twoheadrightarrow b \Rightarrow \exists n,\ b \twoheadrightarrow a^{\bullet^n}$

space of multi-steps ⇒ confluence, cofinality



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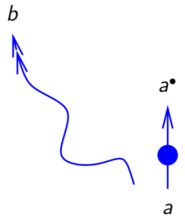
space of multi-steps ⇒ confluence, cofinality



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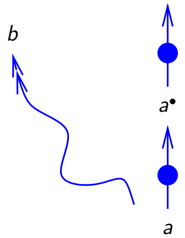

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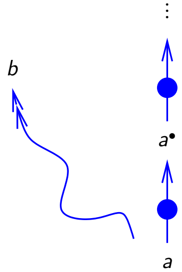
space of multi-steps ⇒ confluence, cofinality



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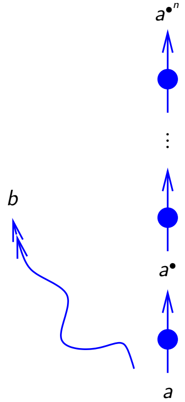

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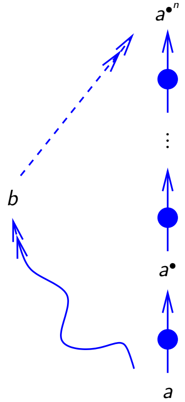

angle property ⇒ $\forall b,\ a \twoheadrightarrow b \Rightarrow \exists n,\ b \twoheadrightarrow a^{\bullet^n}$

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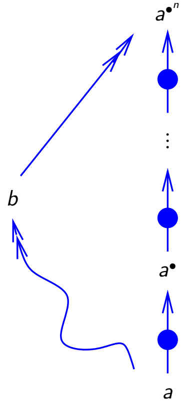
space of multi-steps ⇒ confluence, cofinality



angle property ⇒ cofinality of full multi-step strategy

orthogonalisable

rewriting: simulation by independent steps

orthogonalisable

term rewriting: left-linear and ?

orthogonalisable

term rewriting: left-linear and ?

Examples

orthogonalisable

term rewriting: left-linear and ?

Examples

- lambda-calculus with $\beta\eta$-reduction

$$@(\lambda(x.F(x)), G) \rightarrow F(G)$$
$$\lambda(x.@(F, x)) \rightarrow F$$

orthogonalisable

term rewriting: left-linear and ?

Examples

- lambda-calculus with $\beta\eta$-reduction

$$\begin{array}{rcl} @(\lambda(x.F(x)),G) & \to & F(G) \\ \lambda(x.@(F,x)) & \to & F \end{array}$$

- trivial critical peaks

$$\begin{array}{rcccl} @(F,G) & \leftarrow_\beta & \overline{@(\underline{\lambda(x.@(F,x))},G)} & \to_\eta & @(F,G) \\ \lambda(y.F(y)) & \leftarrow_\eta & \overline{\lambda(x.\underline{@(\lambda(y.F(y)),x)})} & \to_\beta & \lambda(x.F(x)) \end{array}$$

orthogonalisable

term rewriting: left-linear and ?

Examples

- lambda-calculus with $\beta\eta$-reduction

$$\begin{aligned} @(\lambda(x.F(x)), G) &\to F(G) \\ \lambda(x.@(F,x)) &\to F \end{aligned}$$

- weakly orthogonal

$$\begin{array}{cccccc} @(F,G) & \leftarrow_\beta & \overline{@(\underline{\lambda(x.@(F,x))}, G)} & \to_\eta & @(F,G) \\ \lambda(y.F(y)) & \leftarrow_\eta & \overline{\lambda(x.\underline{@(\lambda(y.F(y)),x)})} & \to_\beta & \lambda(x.F(x)) \end{array}$$

- unary integers with rules for successor and predecessor

$$\begin{aligned} S(P(x)) &\to x \\ P(S(x)) &\to x \end{aligned}$$

orthogonalisable

term rewriting: left-linear and ?

Examples

- lambda-calculus with $\beta\eta$-reduction

$$\begin{array}{rcl} @(\lambda(x.F(x)),G) & \to & F(G) \\ \lambda(x.@(F,x)) & \to & F \end{array}$$

- weakly orthogonal

$$\begin{array}{rcccl} @(F,G) & \leftarrow_\beta & \overline{@(\underline{\lambda(x.@(F,x))},G)} & \to_\eta & @(F,G) \\ \lambda(y.F(y)) & \leftarrow_\eta & \overline{\lambda(x.\underline{@(\lambda(y.F(y)),x)})} & \to_\beta & \lambda(x.F(x)) \end{array}$$

- unary integers with rules for successor and predecessor

$$\begin{array}{rcl} S(P(x)) & \to & x \\ P(S(x)) & \to & x \end{array}$$

- trivial criticial peaks

$$\begin{array}{rcccl} S & \leftarrow & \overline{S\,P}\!\!\underline{\;\;\;S} & \to & S \\ P & \leftarrow & \overline{P\,S}\!\!\underline{\;\;\;P} & \to & P \end{array}$$

orthogonalisable

term rewriting: left-linear and ?

Examples

- lambda-calculus with $\beta\eta$-reduction

$$\begin{array}{rcl} @(\lambda(x.F(x)),G) & \rightarrow & F(G) \\ \lambda(x.@(F,x)) & \rightarrow & F \end{array}$$

- weakly orthogonal

$$\begin{array}{rcccl} @(F,G) & \leftarrow_\beta & \overline{@(\underline{\lambda(x.@(F,x))},G)} & \rightarrow_\eta & @(F,G) \\ \lambda(y.F(y)) & \leftarrow_\eta & \overline{\lambda(x.\underline{@(\lambda(y.F(y)),x)})} & \rightarrow_\beta & \lambda(x.F(x)) \end{array}$$

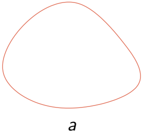
- unary integers with rules for successor and predecessor

$$\begin{array}{rcl} S(P(x)) & \rightarrow & x \\ P(S(x)) & \rightarrow & x \end{array}$$

- weakly orthogonal

$$\begin{array}{rcccl} S & \leftarrow & \overline{S\,\underline{P}}\,\underline{S} & \rightarrow & S \\ P & \leftarrow & \overline{P\,\underline{S}}\,\underline{P} & \rightarrow & P \end{array}$$

orthogonalisation



Definition

- for every object a

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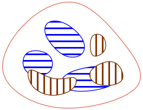
orthogonalisation

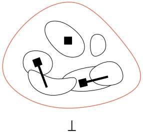


Rdx$_a$

Definition

- for every object a
- its set Rdx$_a$ of redexes

orthogonalisation



Definition

- for every object a
- its set Rdx_a of redexes
- is the (co)domain of a partial function $\perp$ such that

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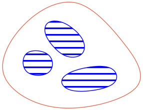
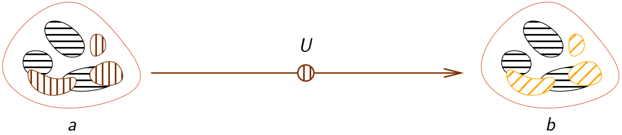


Definition

- for every object a
- its set Rdx_a of redexes
- is the (co)domain of a partial function $\perp$ such that
- its range $Rdx_a^\perp$ is a multi-redex, and

orthogonalisation



Definition

- for every object a
- its set Rdx_a of redexes
- is the (co)domain of a partial function $\perp$ such that
- its range $\mathrm{Rdx}_a^{\perp}$ is a multi-redex, and
- any multi-step $a \multimap_U b$ from a

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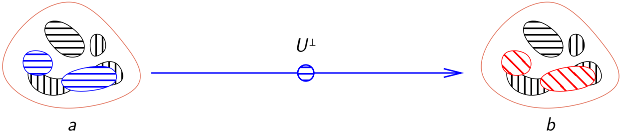
orthogonalisation



Definition

- for every object a
- its set Rdx_a of redexes
- is the (co)domain of a partial function $\perp$ such that
- its range $\mathrm{Rdx}_a^\perp$ is a multi-redex, and
- any multi-step $a \multimap_U b$ from a
- is mapped to an equivalent one $a \multimap_{U^\perp} b$

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orthogonalisation

$P\,S\,P\,P\,S\,S\,P\,S$

Definition

- for every object a
- its set Rdx_a of redexes
- is the (co)domain of a partial function $\perp$ such that
- its range $\mathrm{Rdx}_a^\perp$ is a multi-redex, and
- any multi-step $a \multimap_U b$ from a
- is mapped to an equivalent one $a \multimap_{U^\perp} b$

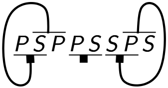
orthogonalisation

$$\underset{1}{\underline{P\,\overset{2}{\overline{S}}}}\,\overline{P}\;\underset{3}{\underline{P\,S}}\;\underset{4}{\underline{S\,\overset{5}{\overline{P}}}}\,\overline{S}$$

Definition

- for every object a
- its set Rdx_a of redexes
- is the (co)domain of a partial function $\perp$ such that
- its range $\mathrm{Rdx}_a^{\perp}$ is a multi-redex, and
- any multi-step $a \multimap\!\!\to_U b$ from a
- is mapped to an equivalent one $a \multimap\!\!\to_{U^{\perp}} b$

orthogonalisation



Definition

- for every object a
- its set Rdx_a of redexes
- is the (co)domain of a partial function $\perp$ such that
- its range $\mathrm{Rdx}_a^{\perp}$ is a multi-redex, and
- any multi-step $a \multimap_U b$ from a
- is mapped to an equivalent one $a \multimap_{U^{\perp}} b$

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orthogonalisation

$\underset{1}{\underline{P\,S}}\,P\,\underset{3}{\underline{P\,S}}\,\underset{4}{\underline{S\,P}}\,S$

Definition

- for every object a
- its set Rdx_a of redexes
- is the (co)domain of a partial function $\perp$ such that
- its range $\mathrm{Rdx}_a^{\perp}$ is a multi-redex, and
- any multi-step $a \multimap_U b$ from a
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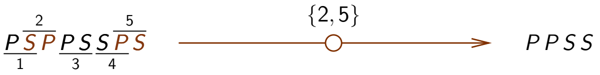
orthogonalisation



Definition

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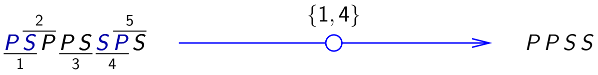

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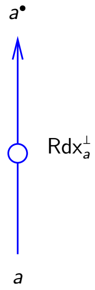
orthogonalisable ⇒ confluence, cofinality

a

$\multimap\!\!\rightarrow$ has angle property: $\forall a$

orthogonalisable ⇒ confluence, cofinality



$\multimap\!\!\to$ has angle property: $\forall a, \exists a^\bullet$

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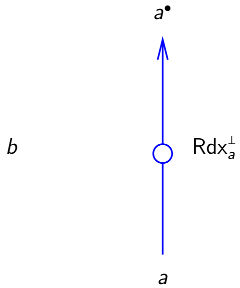

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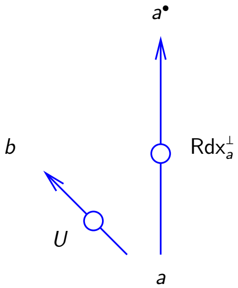

$\multimap\!\!\to$ has angle property: $\forall a, \exists a^\bullet, \forall b,\ a \multimap\!\!\to b$

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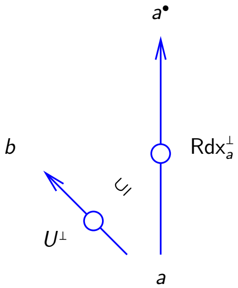

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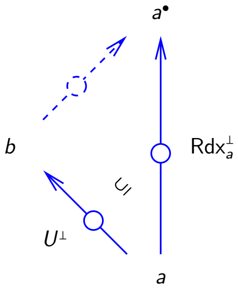

$\multimap\!\!\!\rightarrow$ has angle property: $\forall a,\ \exists a^\bullet,\ \forall b,\ a \multimap\!\!\!\rightarrow b$

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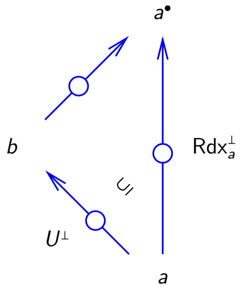

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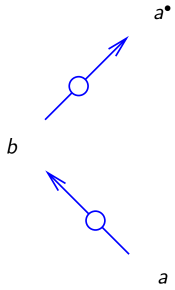

$\multimap\!\!\!\rightarrow$ has angle property: $\forall a, \exists a^\bullet, \forall b, a \multimap\!\!\!\rightarrow b \Rightarrow b \multimap\!\!\!\rightarrow a^\bullet$

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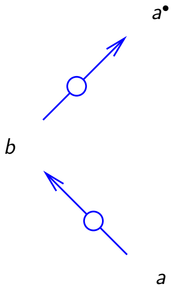

angle property ⇒ confluence, cofinality

orthogonalisation maps

- trivial rules: everywhere undefined

orthogonalisation maps

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- orthogonal rewrite systems: the identity

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$$\underbrace{P\,\overbrace{S}^{}}_{1}\ldots \quad \text{i.e. } \underset{1}{\underline{P\,S}}\;\underset{3}{\underline{P\,S}}\;\underset{5}{\underline{P\,S}} \text{ with overlapping } \overset{2}{\overline{S\,P}}\,,\ \overset{4}{\overline{S\,P}}$$

e.g. (11335) or (55311)

orthogonalisation maps

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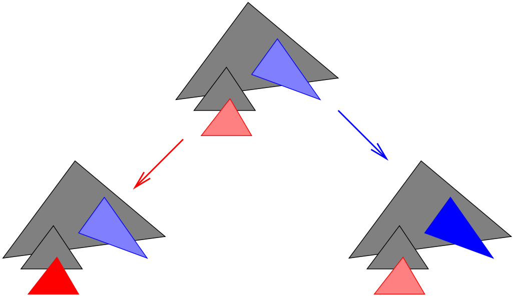

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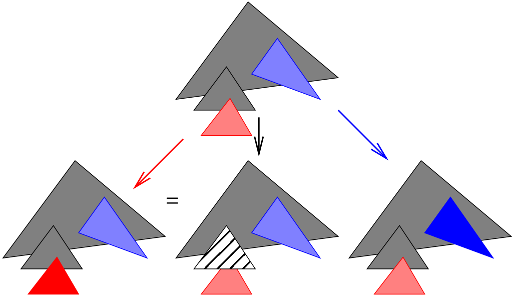

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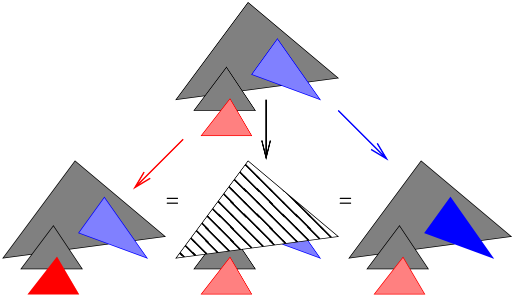

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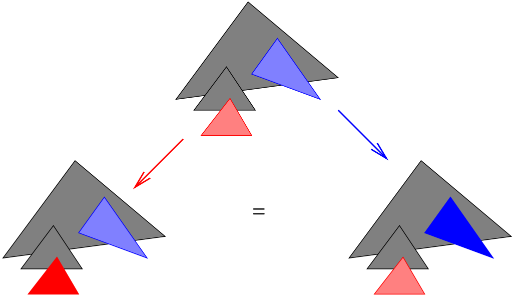

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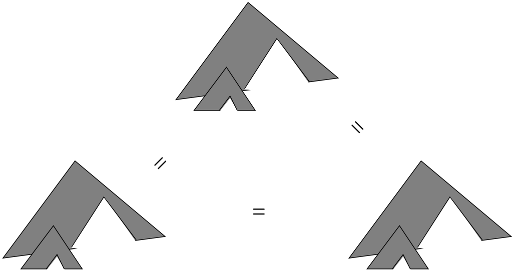

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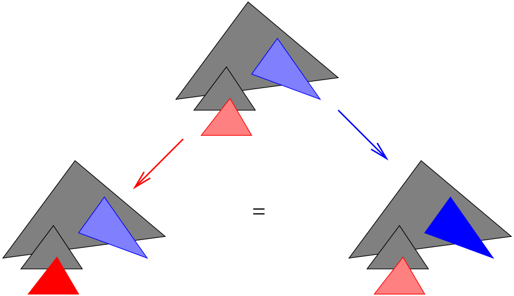

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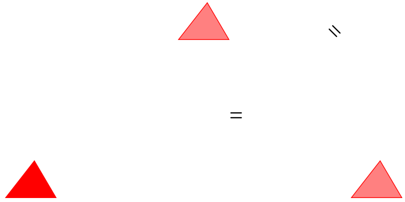

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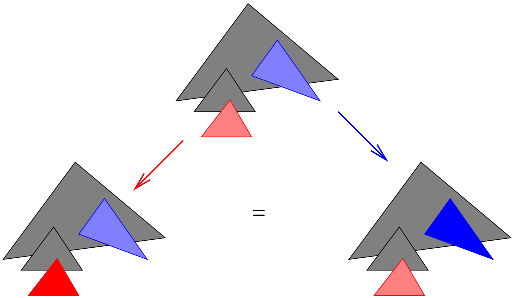

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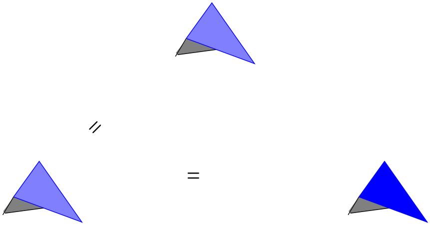

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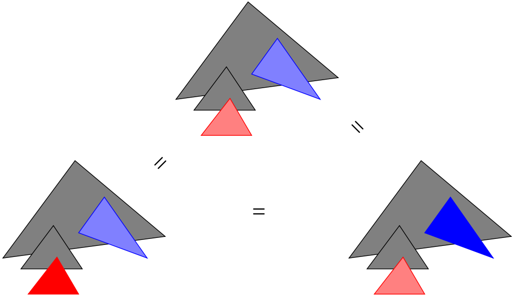

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- critically trivial redexes undefined in

$$\begin{aligned} g(f(a,a)) &\to b \\ f(x,y) &\to f(y,x) \end{aligned}$$

$$b \leftarrow \overline{g(\underline{f(a,a)})} \to g(f(a,a))$$

trivial step as part of a critical peak

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  trivial step as part of a critical peak
- characterise orthogonalisability exactly/decidably?

feebly orthogonal

Definition

rule is redundant if a specialisation of another rule
peak is (ir)redundant if (n)either of its rules is

feebly orthogonal

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peak is (ir)redundant if (n)either of its rules is

$f(g(a)) \to f(a)$ is redundant in presence of $g(x) \to x$

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peak $b \leftarrow a \to c$ is feeble if $|\{b, a, c\}| \leq 2$

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all examples above

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all examples above ... but also

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(non-feeble critical peak(s):

$$g(a) \leftarrow f(a) \to f(b)$$

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(non-feeble critical peak(s):

$g(a) \leftarrow f(a) \to f(b)$ but redundant)

... no coincidence!

orthogonalisable ⇔ feebly orthogonal

Proof.

orthogonalisable ⇔ feebly orthogonal

Proof.

- only if: show every irredundant critical peak feeble

orthogonalisable ⇔ feebly orthogonal

Proof.

- only if: show every irredundant critical peak feeble
  induction on size of source of peak $b \leftarrow_u a \rightarrow_v c$

orthogonalisable ⇔ feebly orthogonal

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- only if: show every irredundant critical peak feeble
  induction on size of source of peak $b \leftarrow_u a \rightarrow_v c$
  interesting orthogonalisation case: $\{u, v\} \mapsto \{u^\perp, v^\perp\}$

orthogonalisable ⇔ feebly orthogonal

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  1. omit redundant redexes from consideration (obvious)

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- if: reduce to the weakly orthogonal case
  1. omit redundant redexes from consideration
  2. map critically trivial redexes to undefined (interesting)

orthogonalisable ⇔ feebly orthogonal

Proof.

- only if: show every irredundant critical peak feeble
  induction on size of source of peak $b \leftarrow_u a \rightarrow_v c$
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- if: reduce to the weakly orthogonal case
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  2. map critically trivial redexes to undefined
     only weakly orthogonal clusters (of trivial peaks) remain;

□

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  4. map redexes in chains to odd ones (as before)

□

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□

novel (higher-order) insights analysing item 2; rest of talk

trivial steps cannot be mapped to undefined

TRS

$$f(x,y) \rightarrow f(y,x)$$
$$a \rightarrow b$$

trivial steps cannot be mapped to undefined

TRS

$$\begin{array}{rcl} f(x,y) & \to & f(y,x) \\ a & \to & b \end{array}$$

orthogonal basis for reduction space from $f(a,a)$:

$$\begin{array}{rcl} \underline{f(a,a)} & \to_u & f(a,a) \\ f(\bar{a},a) & \to_v & f(b,a) \\ f(a,\bar{a}) & \to_w & f(a,b) \end{array}$$

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u extensionally trivial (may map to undefined, in principle)

$$\underline{f(a,a)} \to_u f(a,a)$$
$$f(a,a) \multimap_{\varnothing} f(a,a)$$

trivial steps cannot be mapped to undefined

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u not intensionally trivial (rules out map to undefined)

$$\begin{array}{rcl} \underline{f(\bar{a},a)} & \multimap_{\{u,v\}} & f(a,b) \\ f(\bar{a},a) & \multimap_{\{v\}} & f(b,a) \end{array}$$

critically trivial steps can be mapped to undefined

TRS

$$
\begin{array}{rcl}
f(u,x,y,v) & \to & f(u,y,x,v) \\
f(g(u'),a,a,h(v')) & \to & i(u',v')
\end{array}
$$

critically trivial steps can be mapped to undefined

TRS

$$\begin{array}{rcl} f(u,x,y,v) & \to & f(u,y,x,v) \\ f(g(u'),a,a,h(v')) & \to & i(u',v') \end{array}$$

feeble critical peak

$$f(g(u'),a,a,h(v'')) \leftarrow f(g(u'),a,a,h(v')) \to i(u',v')$$

via substitution

$$\sigma = [u \mapsto g(u'), x \mapsto a, y \mapsto a, v \mapsto h(v')]$$

critically trivial steps can be mapped to undefined

TRS

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- critically trivial step action trivial on open variables u, v

  $$f(u,a,a,v) \leftarrow f(u,a,a,v)$$

critically trivial steps can be mapped to undefined

TRS

$$f(u,x,y,v) \rightarrow f(u,y,x,v)$$
$$f(g(u'),a,a,h(v')) \rightarrow i(u',v')$$

feeble critical peak

$$f(g(u'),a,a,h(v'')) \leftarrow f(g(u'),a,a,h(v')) \rightarrow i(u',v')$$

via substitution

$$\sigma = [u \mapsto g(u'), x \mapsto a, y \mapsto a, v \mapsto h(v')]$$

- critically trivial step action trivial on open variables u, v

  $$f(u,a,a,v) \leftarrow f(u,a,a,v)$$

- other steps' action trivial on closed variables x, y, e.g.

  $$a \rightarrow b$$

  would yield a non-feeble critical peak with other rule

  $$f(g(u'),b,a,h(v'')) \leftarrow f(g(u'),a,a,h(v'))$$

critically trivial steps can be mapped to undefined

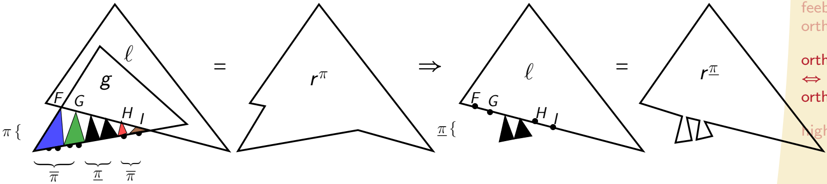
Definition

π is discriminator if each term has unique variable (in range)

critically trivial steps can be mapped to undefined

Definition

π is discriminator if each term has unique variable (in range)



Lemma

if $\ell^{\pi} \to r^{\pi}$ critically trivial (via overlap with $g \to d$)

orthogonalisable

feebly orthogonal

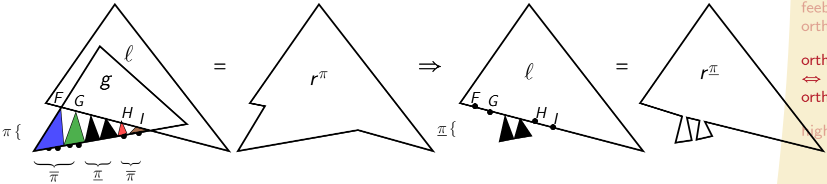
orthogonalisable $\Leftrightarrow$ feebly orthogonal

higher-order

critically trivial steps can be mapped to undefined

Definition

π is discriminator if each term has unique variable (in range)



orthogonalisable

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orthogonalisable ⇔ feebly orthogonal

higher-order

Lemma

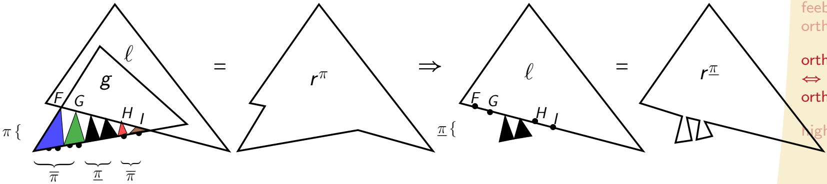
if $\ell^\pi \to r^\pi$ critically trivial

- *π factors as $\overline{\pi} \circ \underline{\pi}$ (obvious)*
  with $\overline{\pi}/\underline{\pi}$ the restriction of π to open/closed terms

critically trivial steps can be mapped to undefined

Definition

π is discriminator if each term has unique variable (in range)



Lemma

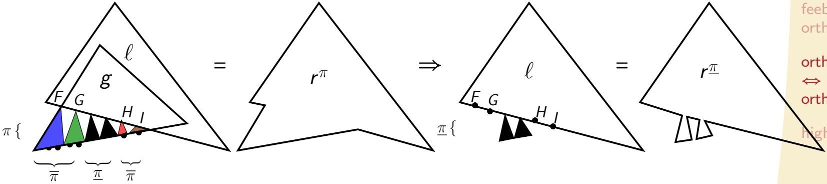
if $\ell^\pi \to r^\pi$ critically trivial

- *π factors as $\overline{\pi} \circ \underline{\pi}$*
  with $\overline{\pi}/\underline{\pi}$ the restriction of π to open/closed terms
- *$\overline{\pi}$ is a discriminator (by left-linearity)*

critically trivial steps can be mapped to undefined

Definition

π is discriminator if each term has unique variable (in range)



Lemma

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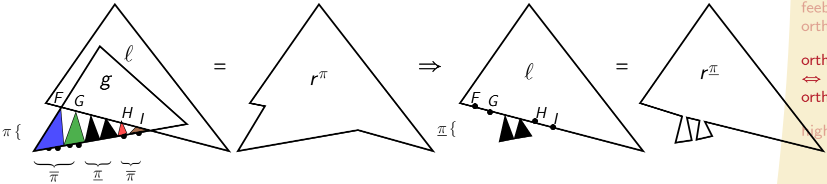
- *π factors as $\overline{\pi} \circ \underline{\pi}$*
  with $\overline{\pi}/\underline{\pi}$ the restriction of π to open/closed terms
- *$\overline{\pi}$ is a discriminator*
- *$\ell^{\underline{\pi}} = r^{\underline{\pi}}$ (by previous item)*

critically trivial steps can be mapped to undefined

Definition

π is discriminator if each term has unique variable (in range)



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  with $\overline{\pi}/\underline{\pi}$ the restriction of π to open/closed terms
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- *$\ell^{\underline{\pi}} = r^{\underline{\pi}}$*

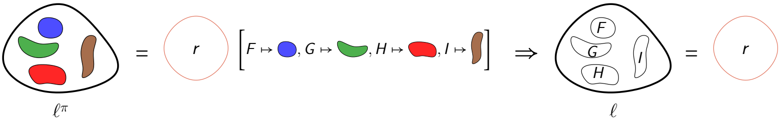
from 2nd to 3rd item based on discrimination lemma

discrimination lemma

substituting a discriminator is reversible

discrimination lemma

substituting a discriminator is reversible



Lemma

for every left-linear rule $\ell \to r$ and discriminator π on the free variables, if $\ell^\pi = r^\pi$, then $\ell = r$.

higher-order

discrimination lemma fails (project vs. imitate)

higher-order

discrimination lemma fails, e.g. for $\pi(F) = x.f(G, x(a))$

$$(x.F(x))^\pi = x.f(G, x(a)) = (x.F(y.x(a)))^\pi$$

higher-order

discrimination lemma fails, e.g. for $\pi(F) = x.f(G, x(a))$

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lhs,π pattern (free variables applied to bound ones)

higher-order

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$$(x.F(x))^\pi = x.f(G, x(a)) = (x.F(y.x(a)))^\pi$$

lhs,π pattern, but active occurrence of bound variable x

bad: patterns may fall apart when substituting for those

higher-order

discrimination lemma fails, e.g. for $\pi(F) = x.f(G, x(a))$

$$(x.F(x))^\pi = x.f(G, x(a)) = (x.F(y.x(a)))^\pi$$

Definition

convex if pattern and no active bound variables

geometric if linear and convex

no variables on path between function symbols in Böhm tree

higher-order

discrimination lemma fails, e.g. for $\pi(F) = x.f(G, x(a))$

$$(x.F(x))^\pi = x.f(G, x(a)) = (x.F(y.x(a)))^\pi$$

Definition

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geometric if linear and convex

Lemma (discrimination)

for every left-geometric rule $\ell \to r$ and geometric discriminator π on the free variables, if $\ell^\pi = r^\pi$, then $\ell^\rho = r^\rho$ for some renaming ρ and geometric substitution $\overline{\pi}$, such that π factors as $\overline{\pi} \circ \rho$

higher-order

discrimination lemma fails, e.g. for $\pi(F) = x.f(G, x(a))$

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Lemma (discrimination)

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Lemma (critically trivial)

if $\ell^\pi \to r^\pi$ critically trivial, with ℓ,π geometric

- π *factors as* $\overline{\pi} \circ \rho \circ \underline{\pi}$
  with $\overline{\pi}/\underline{\pi}$ *the restriction of* π *to open/closed terms*
- $\overline{\pi}$ *is a geometric discriminator,* ρ *a renaming*
- $\ell^{\rho\circ\underline{\pi}} = r^{\rho\circ\underline{\pi}}$

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conclusion

- orthogonalisable as extensional orthogonality

conclusion

- orthogonalisable as extensional orthogonality
- orthogonalisable ⇒ angle/Z-property ⇒ Okui
  angle/Z-property ⇒ confluence and hyper-cofinal strategy

conclusion

- orthogonalisable as extensional orthogonality
- orthogonalisable ⇒ angle/Z-property ⇒ Okui
- orthogonalisable ⇔ feebly orthogonal
  decidable

conclusion

- orthogonalisable as extensional orthogonality
- orthogonalisable $\Rightarrow$ angle/Z-property $\Rightarrow$ Okui
- orthogonalisable $\Leftrightarrow$ feebly orthogonal
- for geometric HRSs (GHRSs); covers extant HRS examples
  left-linear TRS $\subset$ left-linear CRS $\subset$ GHRS $\subset$ left-linear HRS
  second-order matching and higher-order parameters

conclusion

- orthogonalisable as extensional orthogonality
- orthogonalisable $\Rightarrow$ angle/Z-property $\Rightarrow$ Okui
- orthogonalisable $\Leftrightarrow$ feebly orthogonal
- left-linear TRS $\subset$ left-linear CRS $\subset$ GHRS $\subset$ left-linear HRS
- geometric terms well-behaved; closed under
  - substitution
  - application
  - meet
  - join (computed via unification yielding geometric unifier)
  - discrimination

further work

- allow orthogonalisation to map to multi-redexes; characterise

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Definition (orthogonalisation)

function $\perp$ mapping each object a and redex in a, to multi-redex in a, such that $\mathrm{Rdx}_a^{\perp}$ is multi-redex, and any multi-step $a \multimap\!\!\to_U b$ is mapped to equivalent one $a \multimap\!\!\to_{U^\perp} b$.

further work

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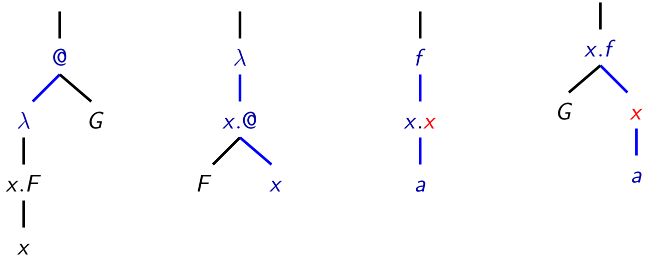
orthogonalisable ⇒ angle/Z-property ⇒ Okui,cofinal

further work

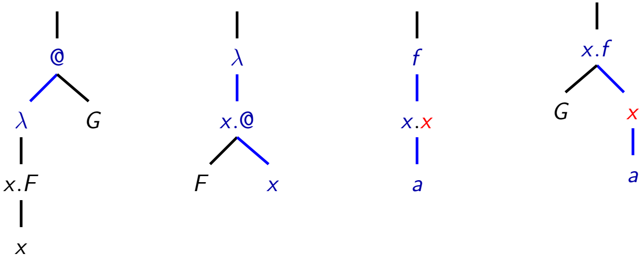
- allow orthogonalisation to map to multi-redexes; characterise
- axiomatize geometricity (GeoRS) allowing geometric proof?

convex



left 2 convex; left-hand sides of β- and η-rules
right 2 not convex; x between f,a; x active, applied to a

convex



left 2 convex; left-hand sides of β- and η-rules
right 2 not convex; x between f,a; x active, applied to a
left-linear PRSs in literature convex

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