Confluence and Higher-Order Critical-Pair-Closing Systems

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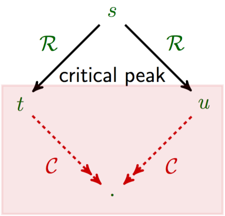
University of Innsbruck

AJSW, September 8th, 2016

Critical-pair-closing systems

Critical-Pair-Closing Systems



$\mathcal{C}$: critical-pair-closing

Definition

$\mathcal{C}$ is critical-pair-closing for $\mathcal{R}$ if $\mathcal{C} \subseteq \mathcal{R}$ and ${}_{\mathcal{R}}\!\leftarrow\!\rtimes\!\rightarrow_{\mathcal{R}} \subseteq \downarrow_{\mathcal{C}}$

Confluence and Critical-Pair-Closing Systems 11/22

Oyamaguchi and Hirokawa, IWC 2014

Confluence criterion based on termination

Criterion based on Termination

suppose $\mathcal{C}$ is critical-pair-closing system for $\mathcal{R}$

Theorem

left-linear TRS $\mathcal{R}$ is confluent if

$\mathcal{C}$ *is terminating and* ${}_{\mathcal{R}}\xleftarrow{>\epsilon} \rtimes \longrightarrow_{\mathcal{R}} \subseteq \mathrel{+\!\!\!\!\twoheadrightarrow}_{\mathcal{C}} \cdot {}_{\mathcal{C}}^{*}\!\!\leftarrow$

Proof

$\to_{\mathcal{C}}^{*} \cdot \mathrel{+\!\!\!\!\twoheadrightarrow}_{\mathcal{R}}$ has diamond property

Corollary

left-linear overlay system $\mathcal{R}$ is confluent if $\mathcal{C}$ is terminating

Oyamaguchi and Hirokawa, 2nd confluence criterion

This talk

- first-order (parallel step) $\rightsquigarrow$ higher-order (multistep)

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- special purpose lemma (O&H) $\rightsquigarrow$ decreasing diagrams

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- special purpose lemma (O&H) $\rightsquigarrow$ decreasing diagrams
- confluence $\rightsquigarrow$ commutation

Higher-order confluence criterion based on termination

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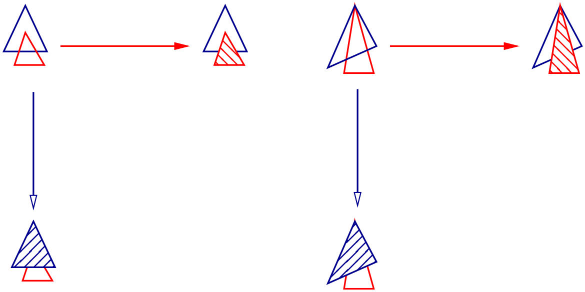
left-linear PRS $\mathcal{R}$ is confluent if for some terminating $\mathcal{C} \subseteq \mathcal{R}$



inner critical peak (and symmetric)

overlay critical peak

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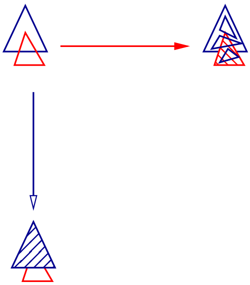

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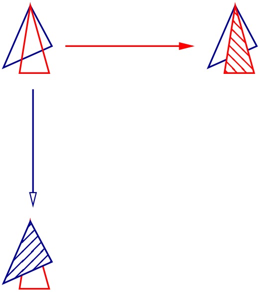

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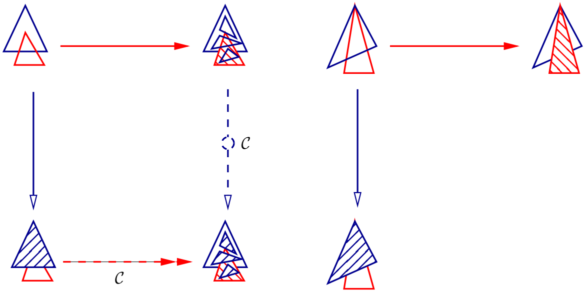

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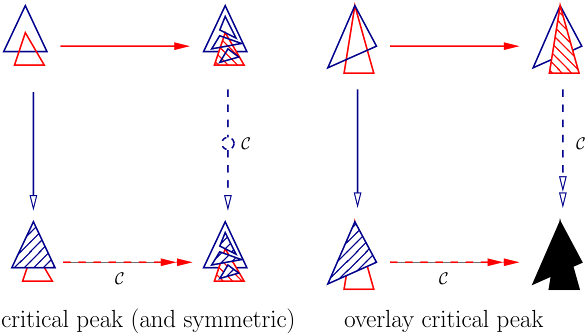
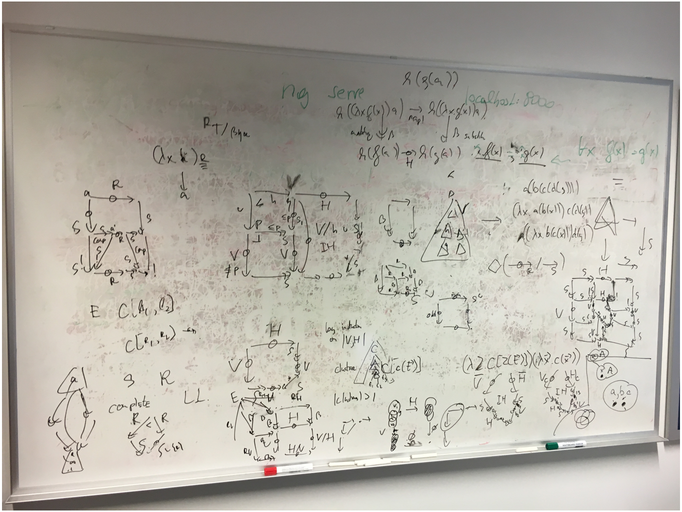


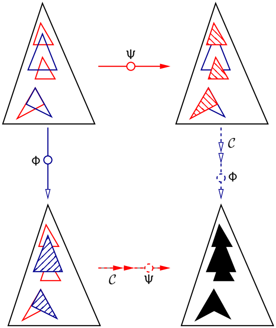
Proof sketch



Date Time: 12 May 2016 14:16:55

Proof

each peak of $\mathcal{R}$-multisteps completed into decreasing diagram



- $\mathcal{C}$-multisteps labelled by source, ordered by $\to_{\mathcal{C}}^{+}$

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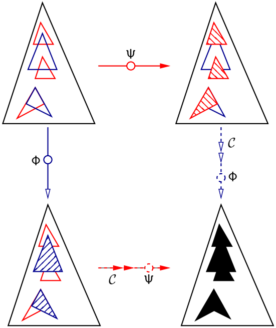

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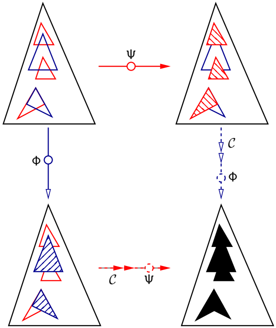

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by well-founded induction #overlap 1st, #clusters 2nd

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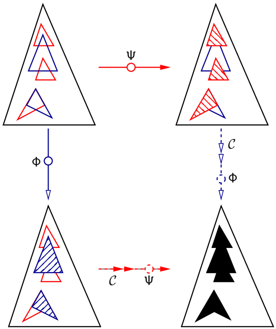

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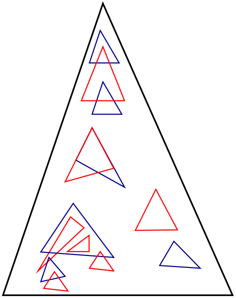
by well-founded induction #overlap 1st, #clusters 2nd
by cases on #clusters 1st, relative positions of patterns 2nd

Overlap and cluster



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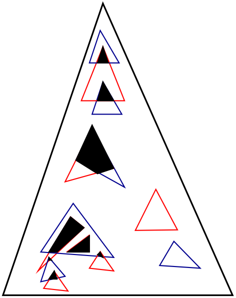
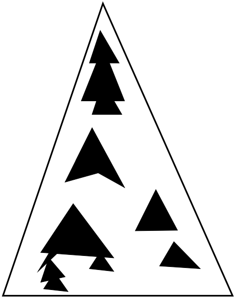


Definition

overlap symbols overlapped by both multisteps

Overlap and cluster



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cluster equivalence closure of overlapping redex-patterns

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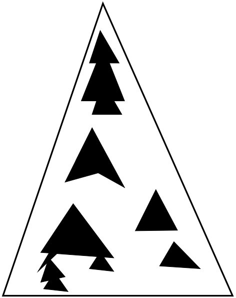

Definition

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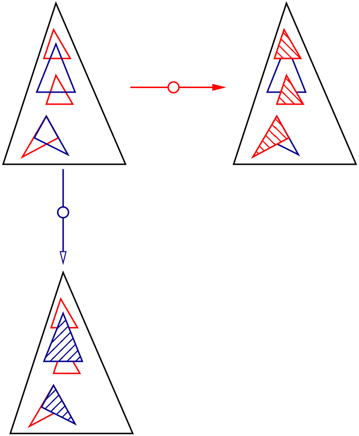
cluster equivalence closure of overlapping redex-patterns

gives rise to patterns (unification of left-hand sides)

Case 1: multi-cluster peak



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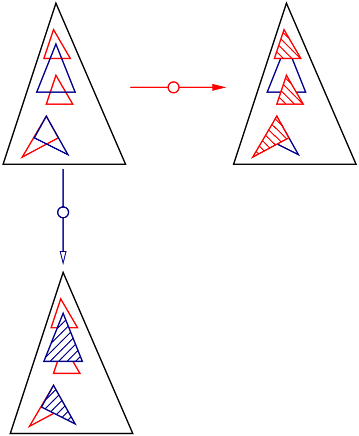
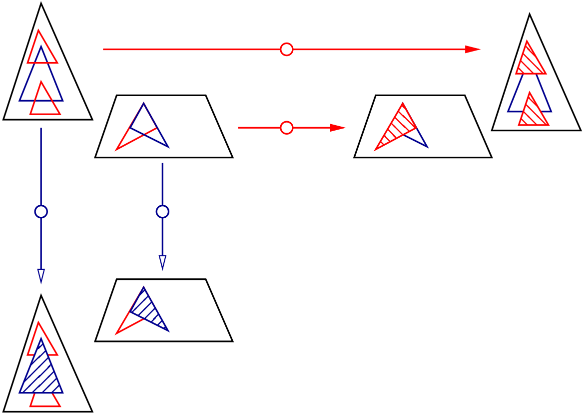


multistep decomposition along cluster

Decomposition



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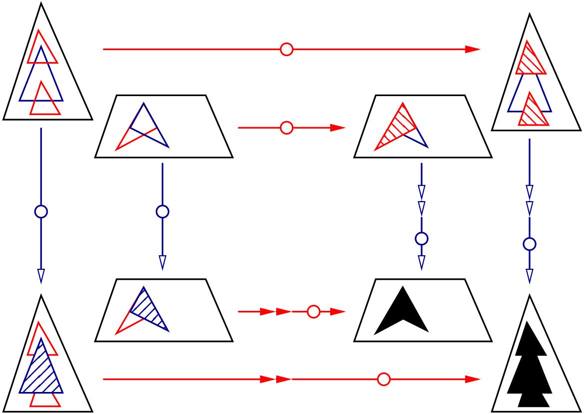
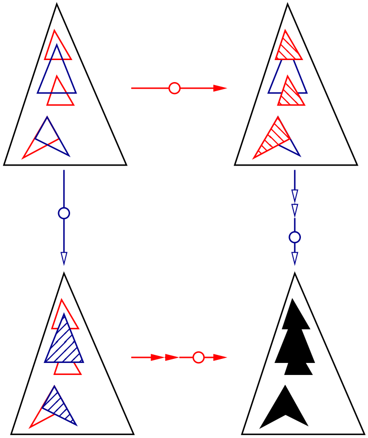


induction hypothesis twice (overlap, clusters)

Recomposition



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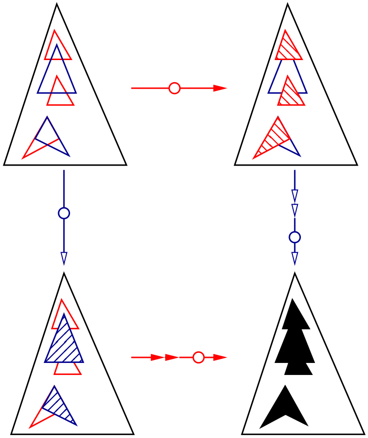
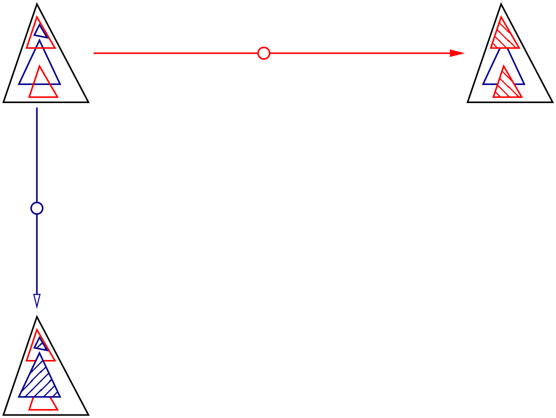


multistep recomposition; preserves decreasingness (choice of labels)

Case 2: inner critical-cluster peak



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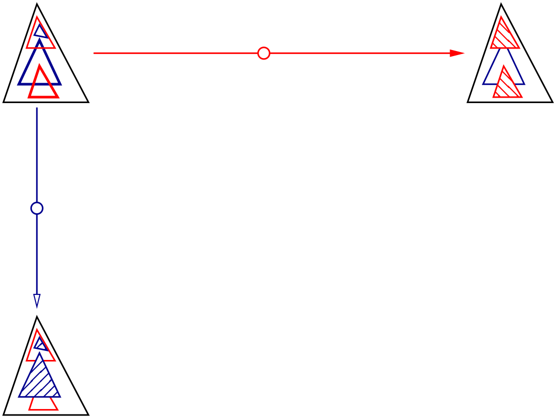
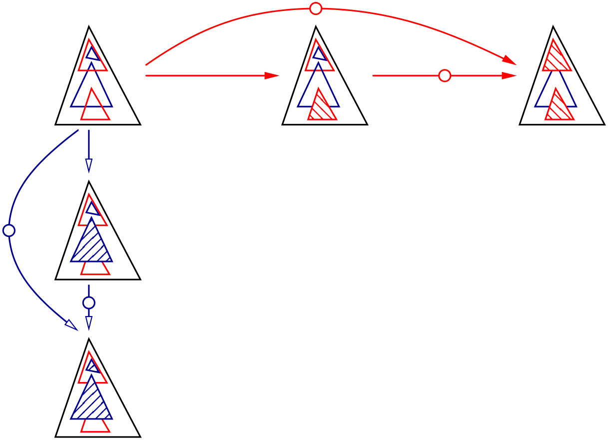


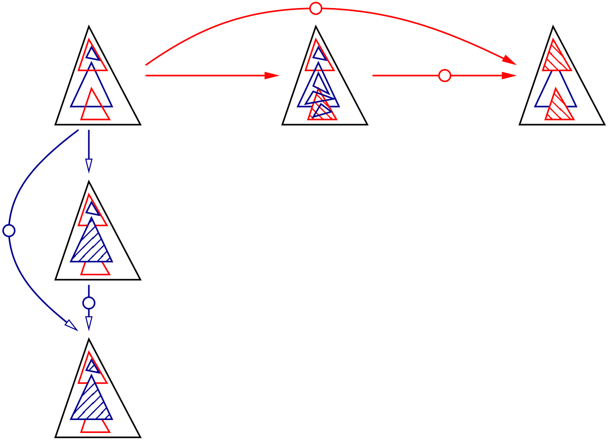
innermost overlap is inner

Inner split



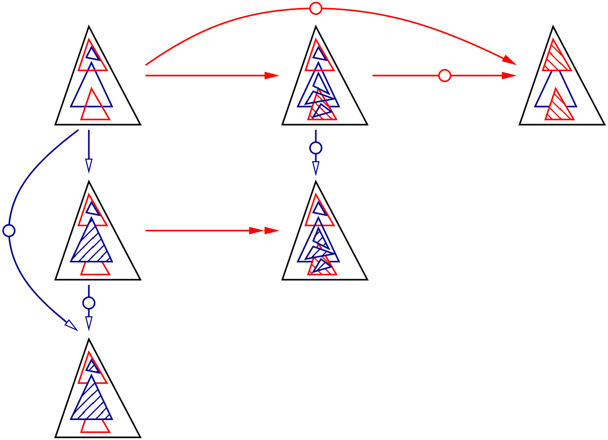
innermost inner critical peak

Inner join



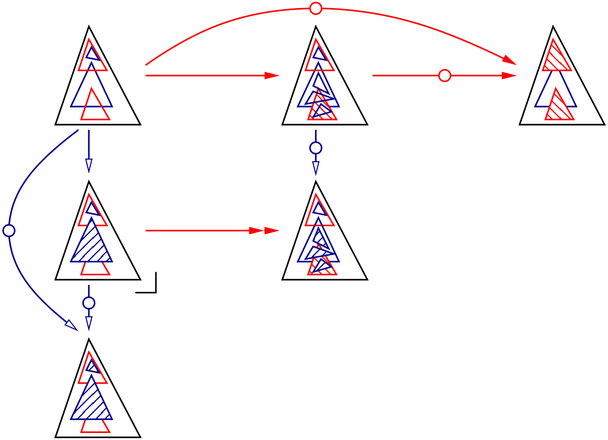
inner critical peak condition

Inner join



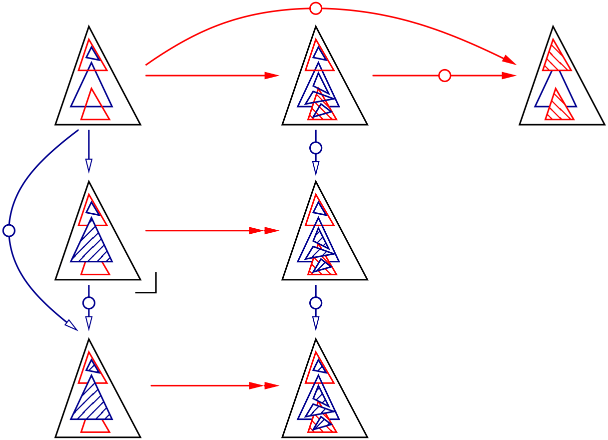
inner critical peak join

Inner join



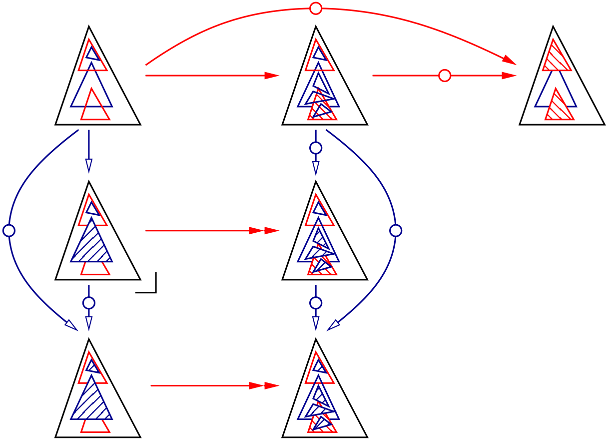
orthogonal

Inner join



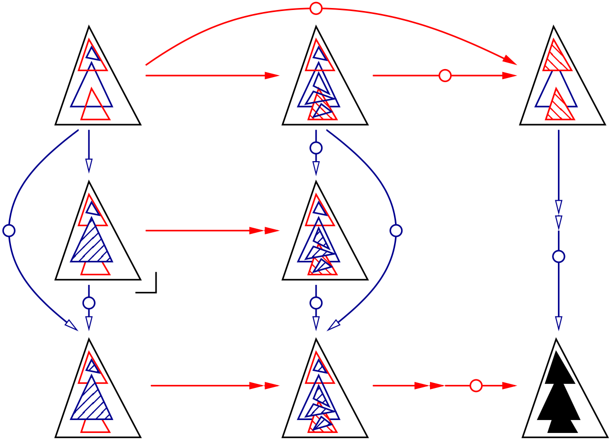
orthogonal projection

Inner join



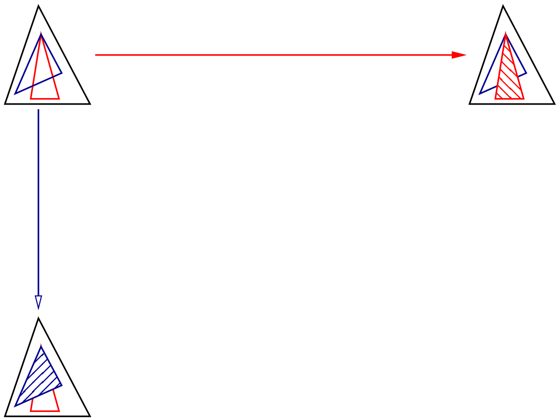
recompose orthogonal multisteps

Inner join



induction hypothesis (overlap); join preserves decreasingness

Case 3: overlay critical-cluster peak



overlay critical peak condition (doubleton cluster)

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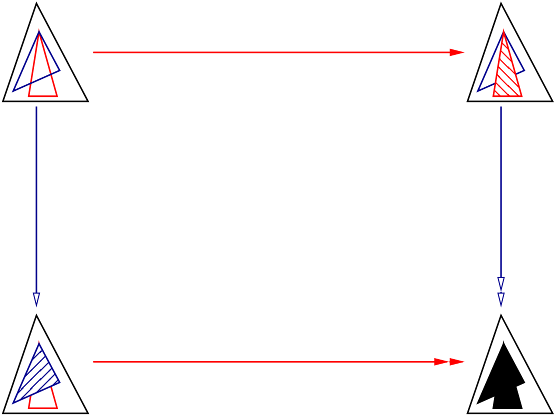

decreasing join

HOT generalisation

Definition

Labelling of PRSs $\mathcal{R}$,$\mathcal{S}$ is HOT if for terminating $\mathcal{C} \subseteq \mathcal{R} \cup \mathcal{S}$

- $\mathcal{C}$-multisteps labelled by source; $\to_{\mathcal{C}}^{+}$

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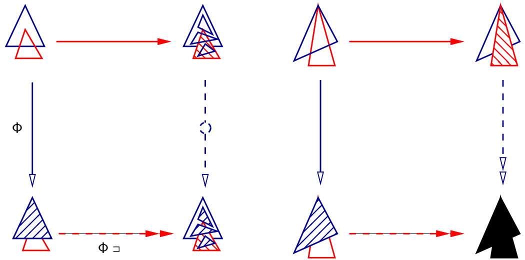
Theorem

left-linear PRSs $\mathcal{R}$,$\mathcal{S}$ commute if critical peaks are HOT-decreasing



inner critical peak (and symmetric)   overlay critical peak

HOT (non-)applications

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7. self-distributivity not covered (different notion of multistep)
   can it be?