The Functional Machine Calculus

confluence via higher-order critical pairs

(Willem Heijltjes & Vincent van Oostrom)[1]

[1]Supported by EPSRC Project EP/R029121/1 Typed lambda-calculi with sharing and unsharing.

The Functional Machine Calculus (FMC)

Two independent modifications to the λ-calculus:

$$M, N \ ::= \quad x \quad | \quad M\,N \quad | \quad \lambda x.M$$

$$M, N \ ::= \ \star \ | \ x.\,M \ | \ [N]a.\,M \ | \ a\langle x\rangle.\,M$$

Sequencing	**Locations**
Split the variable into a unit $\star$ and a variable–with–continuation $x.\,M$	Parameterize abstraction and application in a set of locations $\mathcal{A}$
Encodes **strategies**	Encode **effects**

A simple stack machine/operational semantics (Landin, Krivine)

Stacks: $S \; ::= \; \varepsilon \mid S \cdot M$

$$\frac{}{(\varepsilon\,,\; M)} \quad \frac{(S\,,\; M\,N)}{(S \cdot N\,,\; M)} \quad \frac{(S \cdot N\,,\; \lambda x.\, M)}{(S\,,\; \{N/x\}M)} \quad \frac{(S\,,\; x)}{} \quad \frac{(\varepsilon\,,\; \lambda x.\, M)}{}$$

- **Application**: push
- **Abstraction**: pop and bind to local variable

The poly-λ-calculus

Parameterize application and abstraction in a set of locations $\mathcal{A}$

$$M,N \ ::= \ x \ \mid \ M\,N \ \mid \ \lambda x.M$$

$$M,N \ ::= \ x \ \mid \ [N]a.\,M \ \mid \ a\langle x\rangle.\,M \qquad (a \in \mathcal{A})$$

Embed λ-calculus by a reserved location $\lambda \in \mathcal{A}$ (may omit)

$$\lambda x.\,M \ = \ \lambda\langle x\rangle.\,M \ = \ \langle x\rangle.\,M \qquad M\,N \ = \ [N]\lambda.\,M \ = \ [N].\,M$$

Poly-stack machine/operational semantics

A memory S is a family of stacks: one for every location $a \in \mathcal{A}$.

$$S \;=\; \{S_a \mid a \in \mathcal{A}\} \;=\; S_{a_1}; S_{a_2}; \ldots; S_{a_n}$$

States are pairs (S, M), and transitions are:

$$\frac{(\, S \;;\; S_a \qquad , \; [N]a.\,M \,)}{(\, S \;;\; S_a \cdot N \;, \qquad M \,)} \qquad\qquad \frac{(\, S \;;\; S_a \cdot N \;, \; a\langle x\rangle.\,M \,)}{(\, S \;;\; S_a \qquad , \; \{N/x\}M \,)}$$

Encoding state

A memory cell is modelled by a location $c \in \mathcal{A}$

$$\begin{array}{lrcl} \text{update:} & c := N\,;\, M & = & c\langle\_\rangle.\,[N]c.\,M \\ \text{lookup:} & !c & = & c\langle x\rangle.\,[x]c.\,x \end{array}$$

$$c := N\,;M \;:\quad \frac{\dfrac{(\,S\,;\;\varepsilon_c\cdot P\;,\;c\langle\_\rangle.\,[N]c.\,M\,)}{(\,S\,;\;\varepsilon_c\;,\;[N]c.\,M\,)}}{(\,S\,;\;\varepsilon_c\cdot N\,,\;M\,)}$$

$$!c \;:\quad \frac{\dfrac{(\,S\,;\;\varepsilon_c\cdot N\,,\;c\langle x\rangle.\,[x]c.\,x\,)}{(\,S\,;\;\varepsilon_c\;,\;[N]c.\,N\,)}}{(\,S\,;\;\varepsilon_c\cdot N\,,\;N\,)}$$

β-Reduction

"Skips" stack actions on other locations:

$$[M]a.\,A_1 \ldots A_n.\,a\langle x\rangle.\,N \quad \longrightarrow \quad A_1 \ldots A_n.\,\{M/x\}N$$

where each A_i is an abstraction or application not on location a

Equivalently: "normal" β-reduction

$$[M]a.\,a\langle x\rangle.\,N \longrightarrow \{M/x\}N$$

modulo permutations

$$\begin{aligned}
[M]a.\,[N]b.\,P &\sim [N]b.\,[M]a.\,P \\
a\langle x\rangle.\,[N]b.\,P &\sim [N]b.\,a\langle x\rangle.\,P \quad \text{where } x \notin \mathrm{fv}(N) \\
a\langle x\rangle.\,b\langle y\rangle.\,P &\sim b\langle y\rangle.\,a\langle x\rangle.\,P
\end{aligned}$$

The Functional Machine Calculus

Split the variable into a unit $\star$ and a variable–with–continuation $x.\,M$

$$M, N \;::=\; x \;\mid\; MN \;\mid\; \lambda x.M$$

$$M, N \;::=\; \star \;\mid\; x.\,M \;\mid\; [N]a.\,M \;\mid\; a\langle x\rangle.\,M$$

Some example terms (the trailing $.\,\star$ will be omitted):

$\langle x\rangle.\,[x].\,[x]$ $\quad$ $\langle x\rangle$ $\quad$ $\langle x\rangle.\,[a\langle y\rangle.\,x.\,[y]a]$ $\quad$ $[\mathrm{rnd}\langle x\rangle.\,[x]\mathrm{out}].\,\langle f\rangle.\,f.\,f.\,f$

Nonsense as functions or λ-terms; fine as stack machine instructions!

Composition $N\,;M$ (or $N.\,M$) has unit $\star$ and is capture-avoiding.

$$
\begin{array}{rcll}
\star\,;M &=& M & \\
x.\,N\,;M &=& x.\,(N\,;M) & \\
[P]a.\,N\,;M &=& [P]a.\,(N\,;M) & \\
a\langle x\rangle.\,N\,;M &=& a\langle y\rangle.\,(\{y/x\}N\,;M) & (y \text{ fresh})
\end{array}
$$

Substitution uses composition for the variable case.

$$
\begin{array}{rcll}
\{M/x\}\star &=& \star & \\
\{M/x\}x.\,N &=& M\,;\{M/x\}N & \\
\{M/x\}y.\,N &=& y.\,\{M/x\}N & (x \neq y) \\
\{M/x\}[P]a.\,N &=& [\{M/x\}P]a.\,\{M/x\}N & \\
\{M/x\}a\langle x\rangle.\,N &=& a\langle x\rangle.\,N & \\
\{M/x\}a\langle y\rangle.\,N &=& a\langle z\rangle.\,\{M/x\}\{z/y\}N & (x \neq y, z \text{ fresh})
\end{array}
$$

β-Reduction skips abstractions and applications but not variables,

$$[M]a.\,A_1\,\ldots A_n.\,a\langle x\rangle.\,N \;\longrightarrow\; A_1\,\ldots A_n.\,\{M/x\}N$$

where each A_i is of the form $[P]b$ or $b\langle y\rangle$ with $a \neq b$.

Theorem

β-Reduction in the FMC is confluent.

A rewriter's perspective

dialogue

- P: can you say something about the FMC?

A rewriter's perspective

dialogue

- P: can you say something about the FMC?
- O: what are the objects A and what are the rules P?

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- P: FMC terms and a beta-rule

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- P: just as on the previous slides . . . ?

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analysis and synthesis

1. FMC terms by explicit grammar, with external notion of binding

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1. FMC terms by explicit grammar, with external notion of binding
   $\Rightarrow$ terms as λ-terms over simply typed signature, with λ-binding

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2. FMC open rule schema employing meta-level variables and substitution

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analysis and synthesis

1. FMC terms by explicit grammar, with external notion of binding
   $\Rightarrow$ terms as λ-terms over simply typed signature, with λ-binding
2. FMC open rule schema employing meta-level variables and substitution
   $\Rightarrow$ closed rule schema employing object-level variables and substitution

Higher-order term rewrite systems (Nipkow)

arbitrary signatures

combinatory logic (CL) : term rewrite system (TRS)

$\vdots$ $\vdots$

lambda-calculus (lambda) : higher-order term rewrite system (PRS)

closed under renaming, adding recursion / algebraic rules, etc.

freeness: signature $\Longrightarrow$ terms

simply typed λ-terms modulo $\alpha\beta\eta$ over simply typed signature

implicit grammar (term : simply typed λ-term in long $\beta\eta$-normal form)
internal notion of binding (λ-abstraction)

Higher-order term rewrite systems

Example (addition TRS as a PRS)

- signature $0 : o$ (nullary), $S : o \to o$ (unary), $A : o \to o \to o$ (binary)

Higher-order term rewrite systems

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- signature $0 : o$ (nullary), $S : o \to o$ (unary), $A : o \to o \to o$ (binary)
- rules $\rho : o \to o$ and $\theta : o \to o \to o$, for variables $x, y : o$:

$$\begin{array}{lcl} \rho: \quad \lambda x.A\,x\,0 & \to & \lambda x.x \\ \theta: \lambda xy.A\,x\,(S\,y) & \to & \lambda xy.S\,(A\,x\,y) \end{array}$$

cf. Frege's shift from $\forall x.(t = s)$ to $(\lambda x.t) = (\lambda x.s)$

Higher-order term rewrite systems

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- rules $\rho : o \to o$ and $\theta : o \to o \to o$, for variables $x, y : o$

$$
\begin{array}{lrcl}
\rho: & \lambda x.A(x,0) & \to & \lambda x.x \\
\theta: & \lambda xy.A(x,S(y)) & \to & \lambda xy.S(A(x,y))
\end{array}
$$

with syntactic sugar added

Higher-order term rewrite systems

Example (untyped lambda-beta-eta calculus as a PRS $\mathcal{L}am$)

- signature $\mathsf{abs} : (o \to o) \to o$, $\mathsf{app} : o \to o \to o$

Higher-order term rewrite systems

Example (untyped lambda-beta-eta calculus as a PRS $\mathcal{L}am$)

- signature abs : $(o \rightarrow o) \rightarrow o$, app : $o \rightarrow o \rightarrow o$
- rules eta : $o \rightarrow o$, beta : $(o \rightarrow o) \rightarrow o \rightarrow o$, variables $M : o \rightarrow o$ and $N, K : o$

$$\begin{array}{rrcl} \text{eta:} & \lambda K.\mathsf{abs}\,\lambda x.\mathsf{app}\,K\,x & \rightarrow & \lambda K.K \\ \text{beta:} & \lambda MN.\mathsf{app}\,(\mathsf{abs}\,\lambda x.M\,x)\,N & \rightarrow & \lambda MN.M\,N \end{array}$$

without syntactic sugar; x is parameter to M; K no parameters

Higher-order term rewrite systems

Example (untyped lambda-beta-eta calculus as a PRS $\mathcal{L}am$)

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- rules eta $: o \to o$, beta $: (o \to o) \to o \to o$, variables $M : o \to o$ and $N, K : o$

$$
\begin{array}{rrcl}
\text{eta:} & \lambda K.\text{abs}(\lambda x.\text{app}(K, x)) & \to & \lambda K.K \\
\text{beta:} & \lambda MN.\text{app}(\text{abs}(\lambda x.M(x)), N) & \to & \lambda MN.M(N)
\end{array}
$$

with syntactic sugar

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with syntactic sugar

embedding

PRS $\mathcal{L}am$ 2^{nd}-order since abs : $(o \to o) \to o$.

Higher-order term rewrite systems

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PRS $\mathcal{L}am$ 2^{nd}-order since abs : $(o \to o) \to o$. untyped lambda-calculus embedded in fragment: all variables of type o.

Higher-order term rewrite systems

Example (untyped lambda-beta-eta calculus as a PRS $\mathcal{L}am$)

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embedding

PRS $\mathcal{L}am$ 2^{nd}-order since abs : $(o \to o) \to o$. untyped lambda-calculus embedded in fragment: all variables of type o. $\mathcal{L}am$ orthogonal $\Longrightarrow$ fragment confluent.

FMC substitution

FMC substitution based on auxiliary notion of composition

Composition $N\,;M$ (or $N.\,M$) has unit $\star$ and is capture-avoiding.

$$
\begin{aligned}
\star\;;M &= M \\
x.\,N\,;M &= x.\,(N\,;M) \\
[P]a.\,N\,;M &= [P]a.\,(N\,;M) \\
a\langle x\rangle.\,N\,;M &= a\langle y\rangle.\,(\{y/x\}N\,;M) \quad (y \text{ fresh})
\end{aligned}
$$

Substitution uses composition for the variable case.

$$
\begin{aligned}
\{M/x\}\star &= \star \\
\{M/x\}x.\,N &= M\,;\{M/x\}N \\
\{M/x\}y.\,N &= y.\,\{M/x\}N \quad (x \neq y) \\
\{M/x\}[P]a.\,N &= [\{M/x\}P]a.\,\{M/x\}N \\
\{M/x\}a\langle x\rangle.\,N &= a\langle x\rangle.\,N \\
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\{M/x\}a\langle y\rangle.\,N &= a\langle z\rangle.\,\{M/x\}\{z/y\}N \quad (x \neq y, z \text{ fresh})
\end{aligned}$$

composition $A_1 \ldots A_n.\star\,;M$ is substitution of M for $\star$

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\end{aligned}
$$

represent $A_1 \ldots A_n.\star$ as $\lambda\,\chi.A_1 \ldots A_n.\chi$ for bound variable χ

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Substitution uses composition for the variable case.

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\end{aligned}$$

represent $A_1 \ldots A_n.\star$ as $\lambda\chi.A_1 \ldots A_n.\chi$ for variable χ; $x.\,N$ as application $x\,N$

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\end{aligned}
$$

represent $A_1 \ldots A_n.\star$ as $\lambda\chi.A_1 \ldots A_n.\chi$ for variable χ; increases order!

Higher-order term rewrite systems

Example (the FMC as a PRS $\mathcal{FMC}$)

- signature $\mathsf{abs}_a : ((o \to o) \to o) \to o$, $\mathsf{app}_a : o \to (o \to o) \to o$ for every a

Higher-order term rewrite systems

Example (the FMC as a PRS $\mathcal{FMC}$)

- signature $\mathsf{abs}_a : ((o \to o) \to o) \to o$, $\mathsf{app}_a : o \to (o \to o) \to o$ for every a
- rule schema $\mathsf{beta}_H : \ldots$ for variables N, $\vec{x}$, and x of type $o \to o$ given by:

$$\mathsf{beta}_{a,H}{:}\lambda M\vec{P}N.\mathsf{app}_a(H[\mathsf{abs}_a(\lambda x.M(\vec{x},x))],N) \quad \to \quad \lambda M\vec{P}N.H[M(\vec{x},N)]\lambda MN.M\,N$$

contexts H, given for locations $b \neq a$ by:

$$H \quad ::= \quad \Box \mid \mathsf{app}_b(H,P(\vec{x})) \mid \mathsf{abs}_b(\lambda x.H)$$

$P \in \vec{P}$ fresh variable with as parameters variables bound above

Higher-order term rewrite systems

embedding

$\mathcal{FMC}$ 3rd-order since $\mathsf{abs}_a : ((o \to o) \to o) \to o$

Higher-order term rewrite systems

embedding

FMC embedded by map $\langle\,\rangle$ in fragment $\lambda\chi.S$ of $\mathcal{FMC}$ with

$$S ::= \chi \mid x\,S \mid \mathsf{app}_a(S, \lambda\chi.S) \mid \mathsf{abs}_a(\lambda x.S)$$

- $\star$ maps to χ
- $x.\,M$ maps to $x\,\langle M\rangle$
- $[N]a.\,M$ maps to $\mathsf{app}_a(\langle M\rangle, \lambda\chi.\langle N\rangle)$
- $a\langle x\rangle.\,M$ maps to $\mathsf{abs}_a(\lambda x.\langle M\rangle)$

Higher-order term rewrite systems

embedding

$\mathcal{FMC}$ not orthogonal; (schematic) self-overlaps:

$$\mathsf{app}_a\text{–}\mathsf{app}_b\text{–}\mathsf{abs}_b\text{–}\mathsf{abs}_a$$

$$\mathsf{app}_b\text{–}\mathsf{app}_a\text{–}\mathsf{abs}_b\text{–}\mathsf{abs}_a$$

$\Longrightarrow$ why confluent?

Higher-order term rewrite systems

embedding

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$\Longrightarrow$ why confluent? because overlaps are harmless

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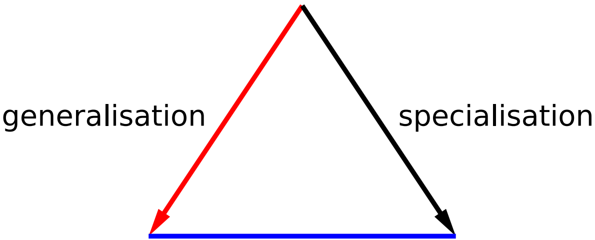
$$\mathsf{app}_b\text{–}\mathsf{app}_a\text{–}\mathsf{abs}_b\text{–}\mathsf{abs}_a$$

$\Longrightarrow$ why confluent? because overlaps are harmless
to express this we need formal notions of overlap, critical peak, step, ...

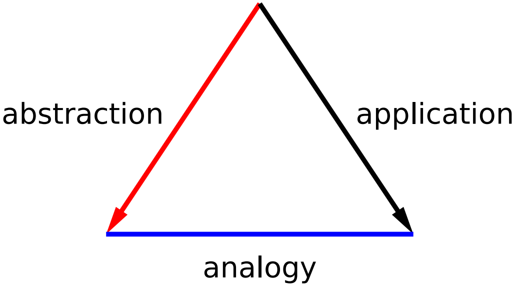
Steps as terms over signature + rules

Pólya's triangle



Steps as terms over signature + rules

Pólya's triangle



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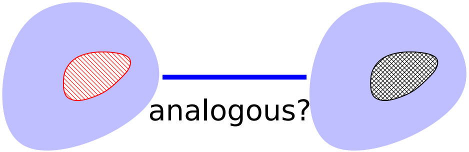
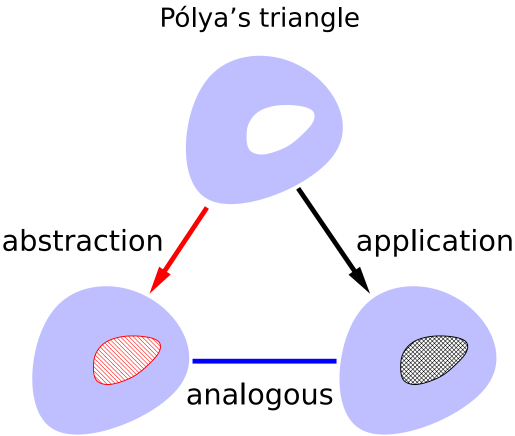


Steps as terms over signature + rules



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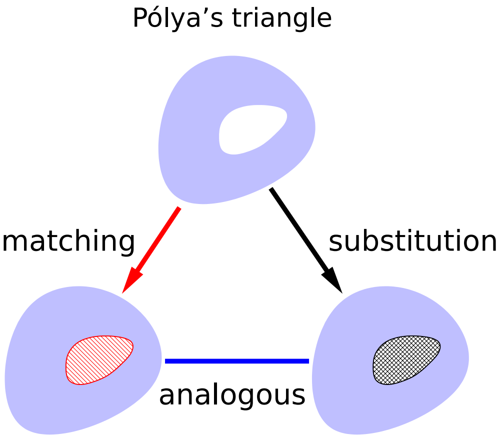
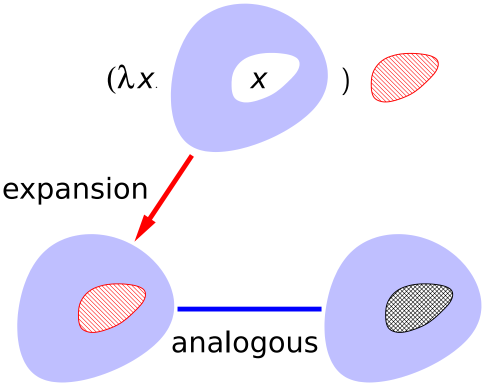


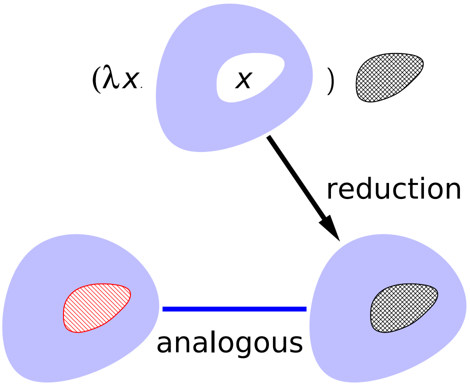
Steps as terms over signature + rules

matching–replacement–substitution rule $\rho : \ell \rightarrow r$ (& van Raamsdonk 1994)



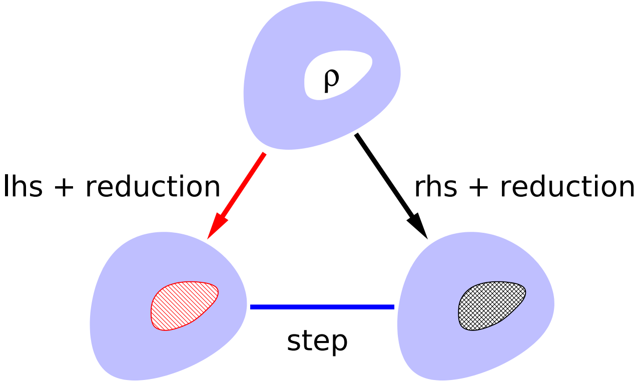
Steps as terms over signature + rules

matching–replacement–substitution rule ρ : $\color{red}{\ell} \to r$



Steps as terms over signature + rules

structured rewrite step for rule $\rho : \ell \to r$ (Terese ⩔ 2003)



Steps as terms over signature + rules

Example (Steps in HRS for addition)

$$\rho\colon \quad \lambda x.A(x,0) \to \lambda x.x$$
$$\theta\colon \lambda xy.A(x,S(y)) \to \lambda xy.S(A(x,y))$$

- $S(\rho(0))$ step from $S((\lambda x.A(x,0))\,0){\downarrow} = S(A(0,0))$ to $S((\lambda x.x)\,0){\downarrow} = S(0)$

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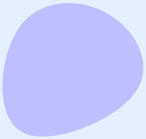
freeness: signature + rules $\Longrightarrow$ multisteps

multistep $\multimap$: simply typed λ-term modulo $\alpha\beta\eta$ over typed signature & rules
source (target) by mapping each rule $\rho : \ell \to r$ in multistep to lhs ℓ (rhs r)
step $\to$ is $\multimap$ restricted to multisteps having one rule (symbol)

Geometric $\Leftrightarrow$ inductive pattern, to define overlap

Idea: allow to carve out well-behaved part, pat $\Longleftrightarrow$ pattern

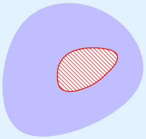
given a term



Geometric $\Leftrightarrow$ inductive pattern, to define overlap

Idea: allow to carve out well-behaved part, pat $\iff$ pattern

select convex set of edges and nodes, a pat P (geometric)



Geometric ⇔ inductive pattern, to define overlap

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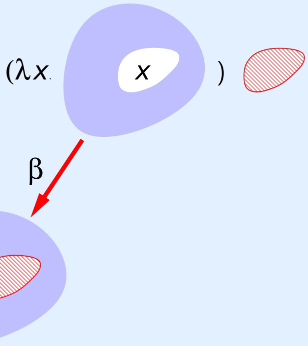
β-expand to occurrence of pattern π (inductive)



Geometric $\Leftrightarrow$ inductive pattern, to define overlap

Definition (pat; geometric)

non-empty set P of positions in tree of λ-term.

- (convex) if $p, q \in P$ then positions on path between p and q in P;

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- (base-fringe) $t_{|p}$ of base type for p root of P or a child not in P of P-position
- (normal) if $t(p)$ is an application and $p \in P$, then left child not λ-position

multipat vector $\vec{P}$ of pairwise disjoint pats

Example

$\{\varepsilon, 1, 11, 12, 121, 122\}$ is pat in $\Delta := \mathrm{app}(\mathrm{abs}(\lambda y.\mathrm{app}(y, y)), \mathrm{abs}(\lambda z.\mathrm{app}(z, z)))$

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Definition (multipattern occurrence; inductive)

positional pattern π is closed term of shape $\lambda \vec{F}.f(\vec{t})$

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$(\lambda\vec{G}.s)\,\vec{\pi}$ multipattern occurrence of $\vec{\pi}$ in $\overline{(\lambda\vec{G}.s)\,\vec{\pi}}$ if s linear in $\vec{G}$, up to permutation of $\vec{\pi}$, $\vec{F}$ (overlining : reduce β-redex and recursively created ones)

Example

lhs $\lambda FS.\mathsf{app}(\mathsf{abs}(\lambda x.F(x)),S)$ of rule beta of $\mathcal{L}am$ is a positional pattern occurring in Δ because $\Delta = \overline{(\lambda G.G(\lambda y.\mathsf{app}(y,y),\mathsf{abs}(\lambda z.\mathsf{app}(z,z))))\,\mathrm{lhs}}$

Geometric vs. inductive patterns

Theorem (geometric vs. inductive)

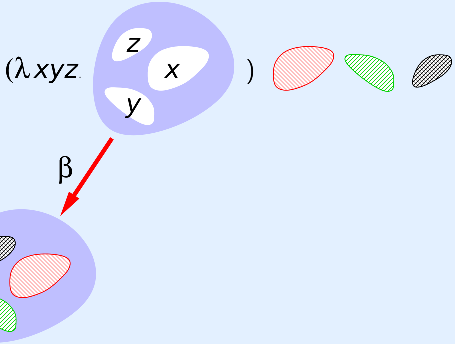
given a λ-term isomorphism between

- *multipats and multipattern-occurrences*

Geometric vs. inductive patterns

isomorphism in a picture

internal positions of patterns bijectively trace to pats



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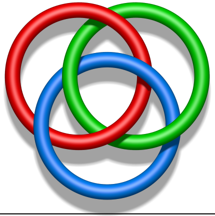
no 3 pairwise disjoint pats not expandable to triplepattern-occurrence (1994)



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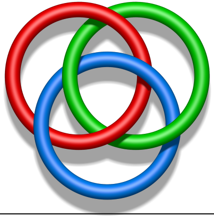

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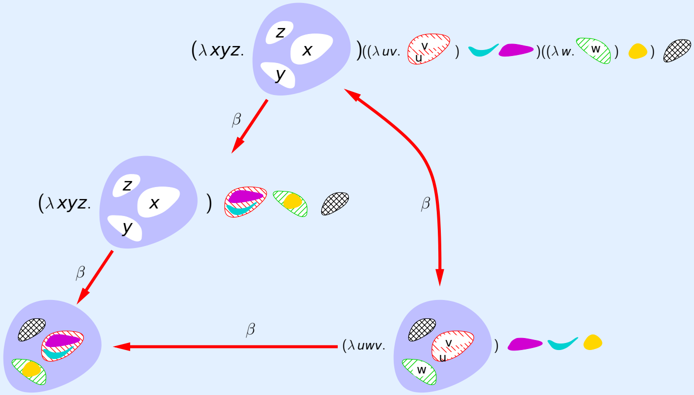
Geometric vs. inductive patterns

refinement isomorphism in a picture



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- *refinement is finite distributive lattice (closed under union, intersection)*

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- redex-patterns orthogonal because there is a multipattern containing them
- redex-patterns overlapping because their pats are (have non-empty intersection) (Hirokawa et al. 2019)

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upshots

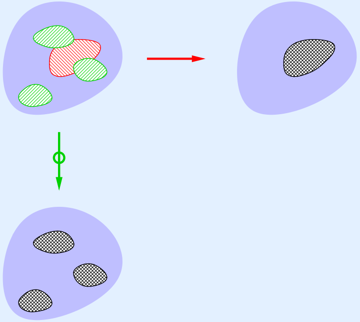
- redex-patterns orthogonal because there is a multipattern containing them
- redex-patterns overlapping because their pats are
- peak is critical if union of pats is the whole source (definition!)

A multi–one critical peak criterion; for TRSs (Okui 1998)

Theorem

$\to$ *is confluent if* $\forall$ *critical multi–one peaks* $b \mathrel{\leftarrow\!\circ\!-} a \to c$, $\exists\, b \twoheadrightarrow d \mathrel{\leftarrow\!\circ\!-} c$

Geometric proof (proof by potatoes).



$\square$

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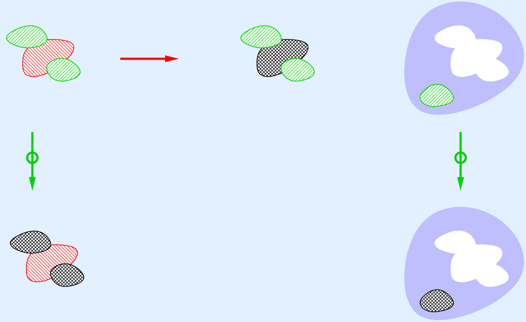

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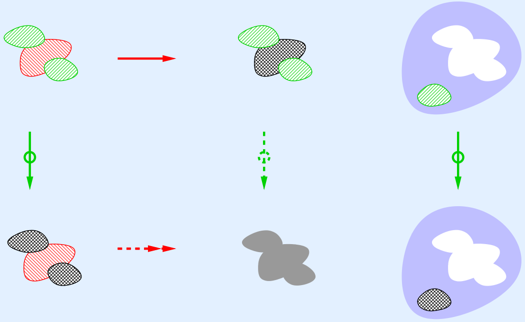

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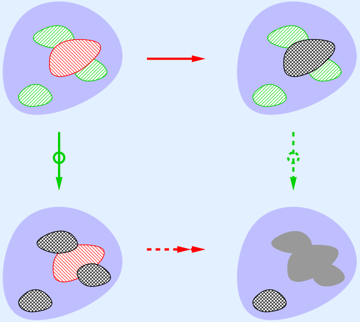

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Inductive proof.

- any overlapping multi–one peak $t \mathrel{\leftarrow\!\!\circ\!\!-} s \rightarrow r$
- decomposes as $(\lambda x.D)\,\hat{t} \mathrel{\leftarrow\!\!\circ\!\!-} (\lambda x.C)\,\hat{s} \rightarrow (\lambda x.C)\,\hat{r}$
  for multi–one critical peak $\hat{t} \mathrel{\leftarrow\!\!\circ\!\!-} \hat{s} \rightarrow \hat{r}$ and multistep $D \mathrel{\leftarrow\!\!\circ\!\!-} C$
- for multi–one critical peak $\hat{t} \mathrel{\leftarrow\!\!\circ\!\!-} \hat{s} \rightarrow \hat{r}$ exists many–multi valley $\hat{t} \twoheadrightarrow \hat{u} \mathrel{\leftarrow\!\!\circ\!\!-} \hat{r}$
- recomposing with multistep $D \mathrel{\leftarrow\!\!\circ\!\!-} C$ yields many–multi valley
  $(\lambda x.D)\,\hat{t} \rightarrow (\lambda x.D)\,\hat{u} \mathrel{\leftarrow\!\!\circ\!\!-} (\lambda x.C)\,\hat{r}$ $\square$

Confluence of $\mathcal{FMC}$

Theorem

$\mathcal{FMC}$ is confluent

Proof.

by checking that all (∞ly many) multi–one critical peaks are many–multi joinable (in fact one–multi) $\square$

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$$\begin{array}{ccc}
-\mathsf{app}_b-\mathsf{app}_a-\mathsf{abs}_b-\mathsf{app}_c-\mathsf{abs}_a-\mathsf{abs}_c- & \to & -\mathsf{app}_b-\mathsf{abs}_b-\mathsf{app}_c-\mathsf{abs}_c- \\
\downarrow & & \downarrow \\
-\mathsf{app}_a-\mathsf{abs}_a- & \to & -
\end{array}$$

$\square$

Further properties of FMC via HRS theory for $\mathcal{FMC}$?

Some rewrite questions and (provisional) answers

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   yes, for typed FMC by Gandy-proof (Barrett, H, McCusker, MFPS 2022)

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5. reductions modulo permutation equivalence a computation category?
   yes, because multisteps $\multimap\!\!\!\!\rightarrow_{\text{beta}}$ constitute residual system

Conclusions

- FMC meta-theory via PRS meta-theory feasible via embedding in $\mathcal{FMC}$

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- FMC semantics via $\mathcal{FMC}$? surely coding of stacks too coarse; linear types?
- rule schema instead of rule? rule pattern is regular language
- work modulo permutation to make beta, eta local? (no modulo theory . . .)

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workshop paper reporting on work in progress; which results? reviewer read precisely (thank you!). reported only minor issues. confluence of FMC proven elsewhere by other means (MFPS 2022). surely result is not in doubt.

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   no standard notion of occurrence of h-o term in literature (programming language theory, proof theory, . . . informal / imprecise / incorrect); used ours (& van Raamsdonk 1994) factoring through HOAS / Church; renders traditional redex-orthogonality-talk obsolete

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   first-order TRSs and second-order frameworks (Klop, Hamana) not closed under abstraction; paraphrasing using contexts, substitutions or ad hoc extending term language possible but awkward (double work)

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   if important, then suggest to also pay (even after contracted has ended)

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   thanks to CL group in Innsbruck for formalising decreasing diagrams converted ($\hookleftarrow$ de Bruijn,Pous), confluence by Z ($\hookleftarrow$ Dehornoy) for λ and CL, modularity of confluence ($\hookleftarrow$ Toyama) via layer framework (Felgenhauer et al.), part of random descent ($\hookleftarrow$ Newman,Toyama), confluence by development-closedness ($\hookleftarrow$ Huet,Toyama) for TRSs, proof terms ($\hookleftarrow$ Meseguer) for left-linear TRSs and to Yamada for part of sub-Birkhoff ($\hookleftarrow$ Plotkin), . . . ?

Opinions to feed your head

1. workshop paper reporting on work in progress
2. sorry; focussed on confluence; more on FMC on Willem's page
3. no standard notion of occurrence of h-o term in literature; used ours
4. use higher-order because closed under abstraction enabling occurrence
5. use inductive $\Leftrightarrow$ geometric view as basis for formalising critical peak criteria
6. conference papers as performance-metric for academic positions is harmful
7. performance-metric proposal: #original results formalised by others

thank you

(return to NL tomorrow night; contact me after at `oostrom@javakade.nl`)

Semantics of / via higher-order term rewriting?

semantics of addition HRS?

$$
\begin{aligned}
\lambda x.A\,x\,0 &\rightarrow_\rho \lambda x.x \\
\lambda xy.A\,x\,(S\,y) &\rightarrow_\theta \lambda xy.S\,(A\,x\,y)
\end{aligned}
$$

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factorise through semantics of substitution; simply typed $\lambda\beta\eta$

- interpret base type o as $\mathbb{N}$ ($[\![o]\!] := \mathbb{N}$), $\tau \to \sigma$ as set $[\![\tau]\!] \Rightarrow [\![\sigma]\!]$ of functions from $[\![\tau]\!]$ to $[\![\sigma]\!]$, function application / abstraction according to their name

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$$\lambda x.A\,x\,0 \;\to_\rho\; \lambda x.x$$
$$\lambda xy.A\,x\,(S\,y) \;\to_\theta\; \lambda xy.S\,(A\,x\,y)$$

factorise through semantics of substitution; simply typed $\lambda\beta\eta$

- interpret each symbol $f : \tau$ as an element of its type $[\![\tau]\!]$, say 0 as $37 \in \mathbb{N}$, S as id $\in \mathbb{N} \Rightarrow \mathbb{N}$, A as first projection $\pi_1 \in \mathbb{N} \Rightarrow \mathbb{N} \Rightarrow \mathbb{N}$

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- interpret rules ρ and θ as equalities

$$\begin{aligned} (n \mapsto n) &= (n \mapsto n) \\ (n, m \mapsto n) &= (n, m \mapsto n) \end{aligned}$$

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- interpret rules ρ and θ as equalities

$$\begin{aligned} (n \mapsto n) &= (n \mapsto n) \\ (n, m \mapsto n) &= (n, m \mapsto n) \end{aligned}$$

of course interpreting as zero, successor, and addition also works

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- interpret rules ρ and θ as equalities

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two, successor, and multiplication gives inequalities $>$ on $\mathbb{N}_{\geq 2}$ (termination)

Semantics of / via higher-order term rewriting?

semantics of untyped lambda-beta-eta HRS?

$$\lambda\,(K).\mathrm{abs}\,\lambda x.\mathrm{app}\,K\,x \quad \to_{\mathrm{eta}} \quad \lambda\,(K).K$$

$$\lambda\,(M\,N).\mathrm{app}\,(\mathrm{abs}\,\lambda x.M\,x)\,N \quad \to_{\mathrm{beta}} \quad \lambda\,(M\,N).M\,N$$

Semantics of / via higher-order term rewriting?

semantics of untyped lambda-beta-eta HRS?

$$\begin{array}{rcl} \lambda\,(K).\mathsf{abs}\,\lambda x.\mathsf{app}\,K\,x & \to_{\mathsf{eta}} & \lambda\,(K).K \\ \lambda\,(M\,N).\mathsf{app}\,(\mathsf{abs}\,\lambda x.M\,x)\,N & \to_{\mathsf{beta}} & \lambda\,(M\,N).M\,N \end{array}$$

factorise through semantics of substitution; simply typed $\lambda\beta\eta$; CCC

- interpret beta and eta-rules in CCC (cf. Koymans):

$$\begin{array}{rcl} @ \circ \langle [\![\mathsf{abs}]\!] \circ \langle\,\rangle, @ \circ \langle [\![\mathsf{app}]\!] \circ \langle\,\rangle, \mathsf{id}\rangle\rangle & = & \mathsf{id} \\ @ \circ \langle [\![\mathsf{app}]\!] \circ \langle\,\rangle, @ \circ \langle [\![\mathsf{abs}]\!] \circ \langle\,\rangle, \mathsf{id}\rangle\rangle & = & \mathsf{id} \end{array}$$

Semantics of / via higher-order term rewriting?

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- for set / functions: $[\![\mathsf{abs}]\!] \circ [\![\mathsf{app}]\!] = \mathsf{id}$ on D and $[\![\mathsf{app}]\!] \circ [\![\mathsf{abs}]\!] = \mathsf{id}$ on $D \Rightarrow D$