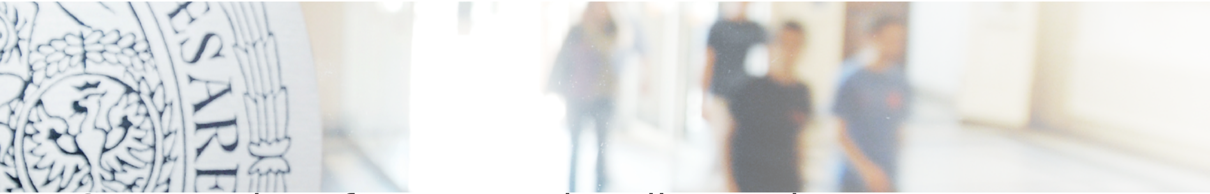
universität innsbruck



Symmetries of commutation diamonds

Vincent van Oostrom

http://cl-informatik.uibk.ac.at

Tiling the plane

Tiling peaks

- with diamonds
- with right-faceted diamonds
- with multi-faceted diamonds

Making diamonds decreasing

- β,η-factorisation
- spine,vertebrae-factorisation
- self-commutation of some term rewrite system

Take-aways

tiling the plane (Hao Wang 1961)

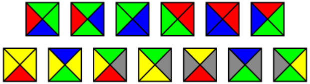
decision problem

given set of tiles, can it tile the plane?

tiling the plane

decision problem

given set of tiles, can it tile the plane?



tiling the plane

decision problem

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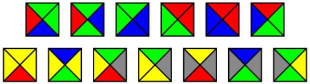


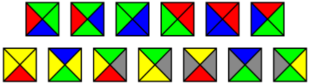
conjecture

any solution will be periodic, so decidable

tiling the plane

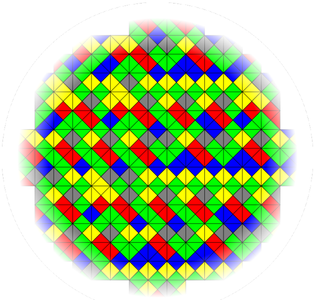
decision problem

given set of tiles, can it tile the plane?



refutation

no, aperiodic tiling; simulate Turing machine (halting iff plane not tiled; Berger 1966)



diamonds (Newman 1942, Hindley 1964, Rosen 1973)

commutation problem (${}_1{\twoheadleftarrow} \cdot {\twoheadrightarrow}_2 \subseteq {\twoheadrightarrow}_2 \cdot {}_1{\twoheadleftarrow}$?)

for term rewrite systems $\mathcal{T}_1$ and $\mathcal{T}_2$

- $\mathcal{T}_1 = \{a \to b\}$
- $\mathcal{T}_2 = \{f(x) \to g(f(x)), f(x) \to h(x)\}$

$\twoheadrightarrow$ is repetition of $\to$; problem equivalent to Church–Rosser $({}_1{\leftarrow} \cup {\to}_2)^* \subseteq {\twoheadrightarrow}_2 \cdot {}_1{\twoheadleftarrow}$

diamonds

commutation problem $({}_1\twoheadleftarrow \cdot \twoheadrightarrow_2 \subseteq \twoheadrightarrow_2 \cdot {}_1\twoheadleftarrow?)$

for term rewrite systems $\mathcal{T}_1$ and $\mathcal{T}_2$

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commutation diamond $({}_1\leftarrow \cdot \to_2 \subseteq \to_2 \cdot {}_1\leftarrow)$

no critical peaks between $\mathcal{T}_1, \mathcal{T}_2$, and for non-critical peaks:

- $\leftarrow \cdot \to \subseteq \to \cdot \leftarrow$ (rules linear)
- $\leftarrow \cdot \to \subseteq \to \cdot \leftarrow$ (rules linear)

diamonds

commutation problem (${}_1\twoheadleftarrow \cdot \twoheadrightarrow_2 \subseteq \twoheadrightarrow_2 \cdot {}_1\twoheadleftarrow$?)

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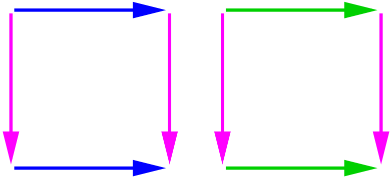
commutation diamond (${}_1\leftarrow \cdot \rightarrow_2 \subseteq \rightarrow_2 \cdot {}_1\leftarrow$)

no critical peaks between $\mathcal{T}_1, \mathcal{T}_2$, and for non-critical peaks:

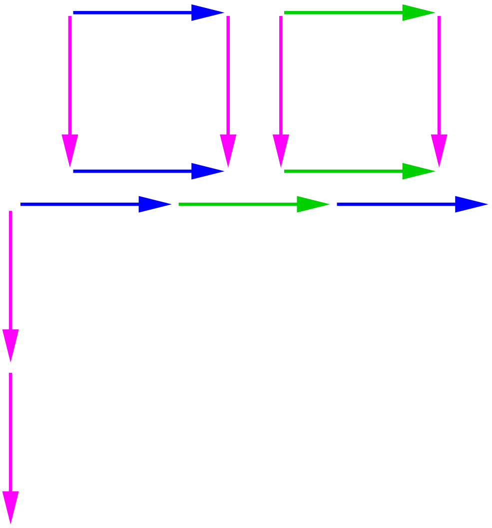
- $\leftarrow \cdot \rightarrow \subseteq \rightarrow \cdot \leftarrow$
- $\leftarrow \cdot \rightarrow \subseteq \rightarrow \cdot \leftarrow$

more precisely ${}_1^n\leftarrow \cdot \rightarrow_2^m \subseteq \rightarrow_2^m \cdot {}_1^n\leftarrow$
and random descent (reductions to common reduct have same length)

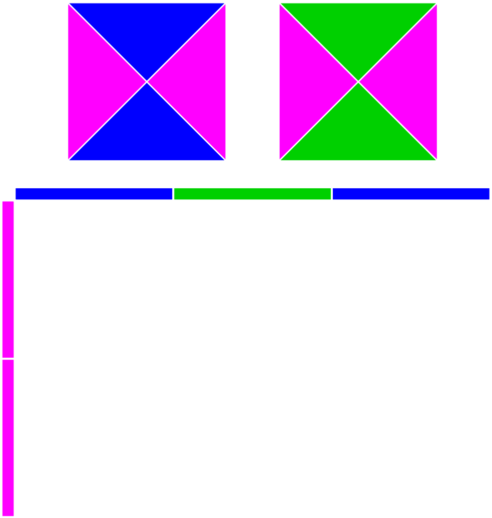
diamonds



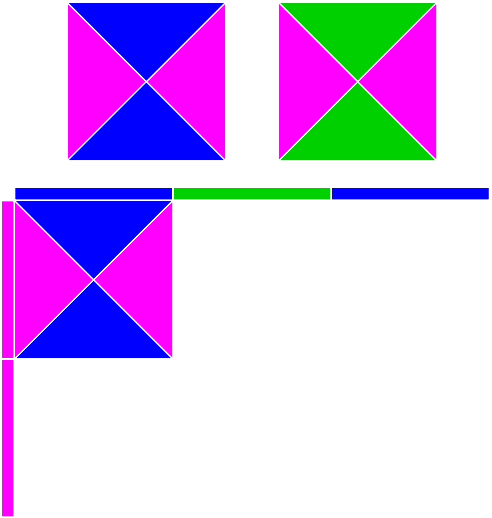
diamonds



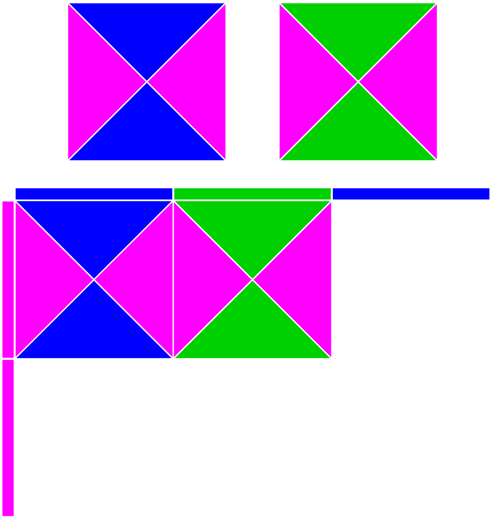
diamonds



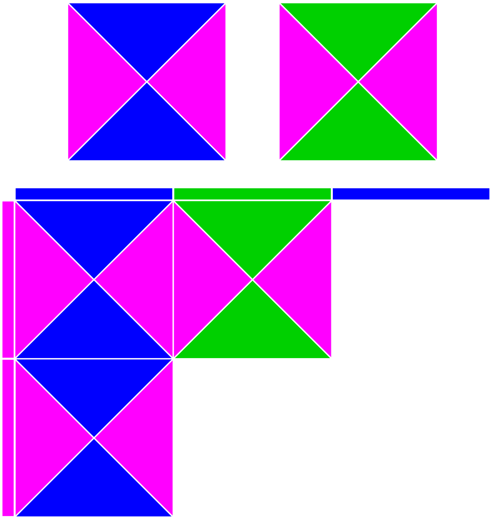
diamonds



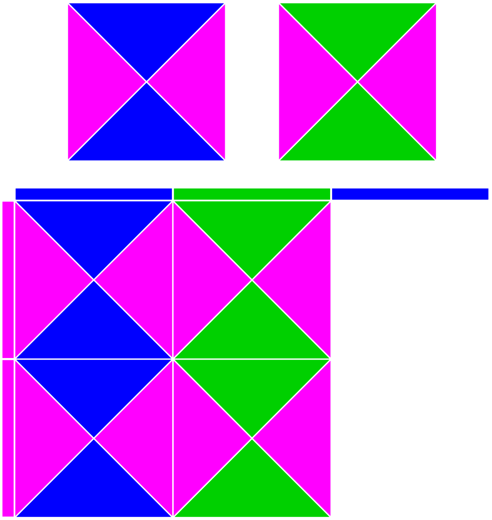
diamonds



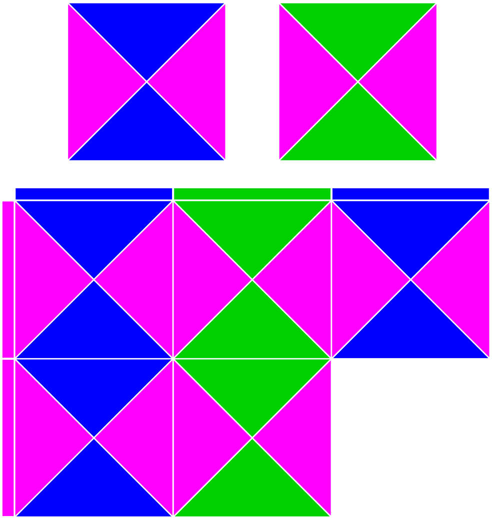
diamonds



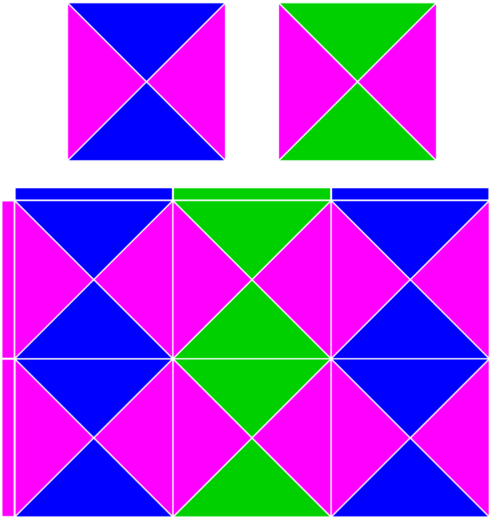
diamonds



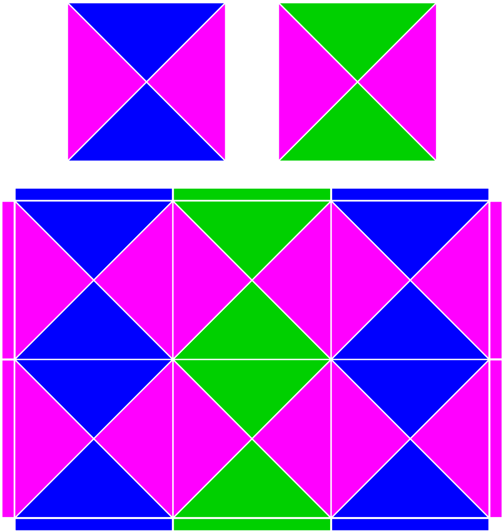
diamonds



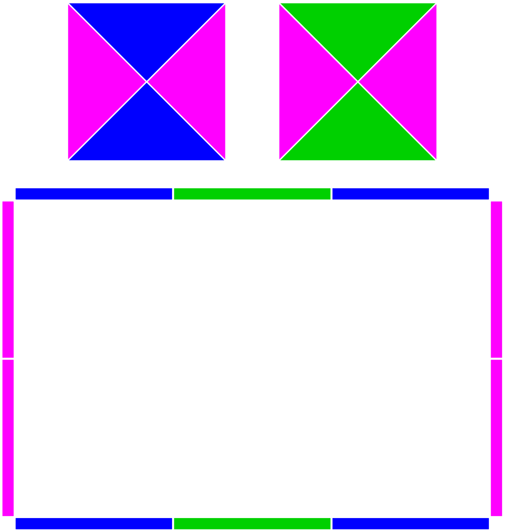
diamonds



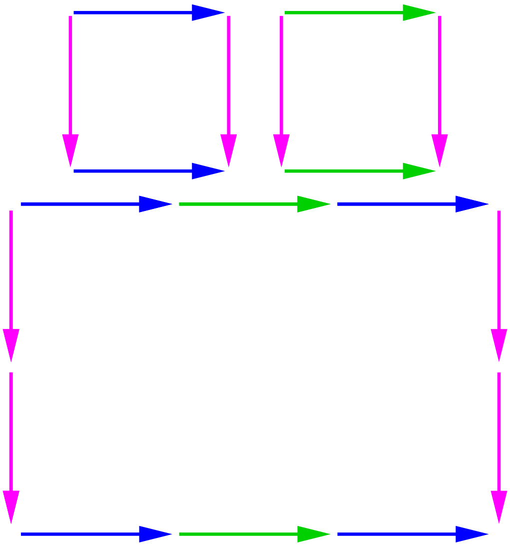
diamonds



diamonds



diamonds



diamonds

factorisation problem $(\twoheadrightarrow_1 \cdot \twoheadrightarrow_2 \subseteq \twoheadrightarrow_2 \cdot \twoheadrightarrow_1?)$

for term rewrite systems $\mathcal{T}_1$ and $\mathcal{T}_2$

- $\mathcal{T}_1 = \{a \to b\}$
- $\mathcal{T}_2 = \{f(x) \to g(f(x)), f(x) \to h(x)\}$

diamonds

factorisation problem ($\twoheadrightarrow_1 \cdot \twoheadrightarrow_2 \subseteq \twoheadrightarrow_2 \cdot \twoheadrightarrow_1$?)

for term rewrite systems $\mathcal{T}_1$ and $\mathcal{T}_2$

- $\mathcal{T}_1 = \{a \to b\}$
- $\mathcal{T}_2 = \{f(x) \to g(f(x)), f(x) \to h(x)\}$

a.k.a. preponement, postponement, commutation over, separation; problem equivalent to $(\to_1 \cup \to_2)^* \subseteq \twoheadrightarrow_2 \cdot \twoheadrightarrow_1$; note $\to_2, \to_1$-factorisation is ${}_1\!\leftarrow, \to_2$-commutation

factorisation diamond ($\to_1 \cdot \to_2 \subseteq \to_2 \cdot \to_1$)

no critical peaks between $\mathcal{T}_1^{-1}, \mathcal{T}_2$, and for non-critical peaks:

- $\to \cdot \to \subseteq \to \cdot \to$ (rules linear)
- $\to \cdot \to \subseteq \to \cdot \to$ (rules linear)

diamonds

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factorisation diamond $(\to_1 \cdot \to_2 \subseteq \to_2 \cdot \to_1)$

no critical peaks between $\mathcal{T}_1^{-1}, \mathcal{T}_2$, and for non-critical peaks:

- $\to \cdot \to \subseteq \to \cdot \to$
- $\to \cdot \to \subseteq \to \cdot \to$

no critical peaks between $\mathcal{T}_1^{-1}, \mathcal{T}_2$ means no overlap between rhss of $\mathcal{T}_1$ and lhss of $\mathcal{T}_2$: $\mathcal{T}_1$ does not create $\mathcal{T}_2$. commutation is factorisation up to symmetry.

diamonds

factorisation problem ($\twoheadrightarrow_1 \cdot \twoheadrightarrow_2 \subseteq \twoheadrightarrow_2 \cdot \twoheadrightarrow_1$?)

for term rewrite systems $\mathcal{T}_1$ and $\mathcal{T}_2$

- $\mathcal{T}_1 = \{a \to b\}$
- $\mathcal{T}_2 = \{f(x) \to g(f(x)), f(x) \to h(x)\}$

a.k.a. preponement, postponement, commutation over, separation; problem equivalent to $(\to_1 \cup \to_2)^* \subseteq \twoheadrightarrow_2 \cdot \twoheadrightarrow_1$; note $\to_2, \to_1$-factorisation is ${}_1\!\leftarrow, \to_2$-commutation

factorisation diamond ($\to_1 \cdot \to_2 \subseteq \to_2 \cdot \to_1$)

no critical peaks between $\mathcal{T}_1^{-1}, \mathcal{T}_2$, and for non-critical peaks:

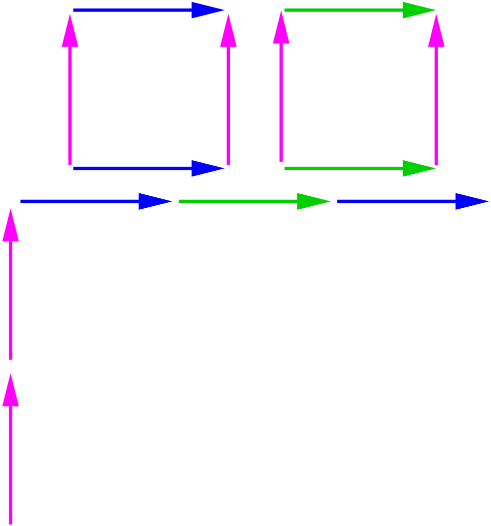
- $\to \cdot \to \subseteq \to \cdot \to$
- $\to \cdot \to \subseteq \to \cdot \to$

commutation and factorisation of given rewrite system independent

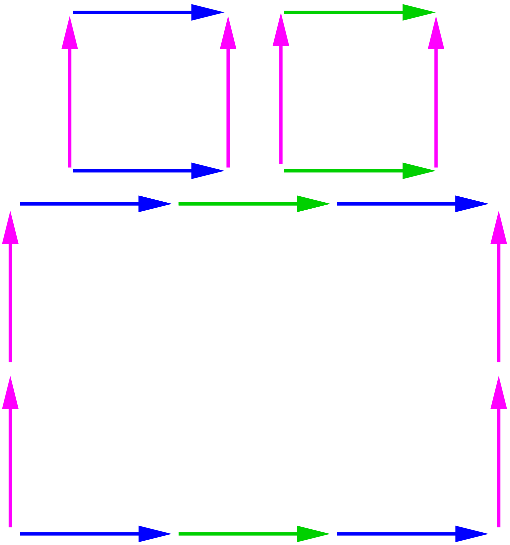
$a \to b$, $a \to c$ has $\to, \to$-factorisation, no $\to, \to$-commutation

$b \to a$, $a \to c$ has $\to, \to$-commutation, no $\to, \to$-factorisation

diamonds



diamonds



right-faceted diamonds (Hindley 1964, Huet 1978)

commutation problem (${}_1\twoheadleftarrow \cdot \twoheadrightarrow_2 \subseteq \twoheadrightarrow_2 \cdot {}_1\twoheadleftarrow$?)

for term rewrite systems $\mathcal{T}_1$ and $\mathcal{T}_2$

- $\mathcal{T}_1 = \{\lambda y.P\,y \to P\}$ (η-reduction in λ-calculus, as HRS rule)
- $\mathcal{T}_2 = \{(\lambda x.M(x))\,N \to M(N)\}$ (β-reduction in λ-calculus, as HRS rule)

β is replicating, not linear; moreover 2 critical peaks; no diamonds

right-faceted diamonds

commutation problem (${}_1{\twoheadleftarrow} \cdot {\twoheadrightarrow}_2 \subseteq {\twoheadrightarrow}_2 \cdot {}_1{\twoheadleftarrow}$?)

for term rewrite systems $\mathcal{T}_1$ and $\mathcal{T}_2$

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- $\mathcal{T}_2 = \{(\lambda x.M(x))\, N \to M(N)\}$

commutation right-faceted diamond (${}_1{\leftarrow} \cdot {\to}_2 \subseteq {\to}_2 \cdot {}_1{\twoheadleftarrow}$)

- $\lambda x.M(x) \leftarrow \lambda y.(\lambda x.M(x))\, y \to \lambda y.M(y)$ (trivial critical peak, up to α)
- $P\, N \leftarrow (\lambda y.P\, y)\, N \to P\, N$ (trivial critical peak)
- $\leftarrow \cdot \to \subseteq \to \cdot \twoheadleftarrow$ (non-critical peaks; η linear, β replicating)

right-faceted diamonds

commutation problem ($_1\twoheadleftarrow \cdot \twoheadrightarrow_2 \subseteq \twoheadrightarrow_2 \cdot {}_1\twoheadleftarrow ?$)

for term rewrite systems $\mathcal{T}_1$ and $\mathcal{T}_2$

- $\mathcal{T}_1 = \{\lambda y.P\,y \to P\}$
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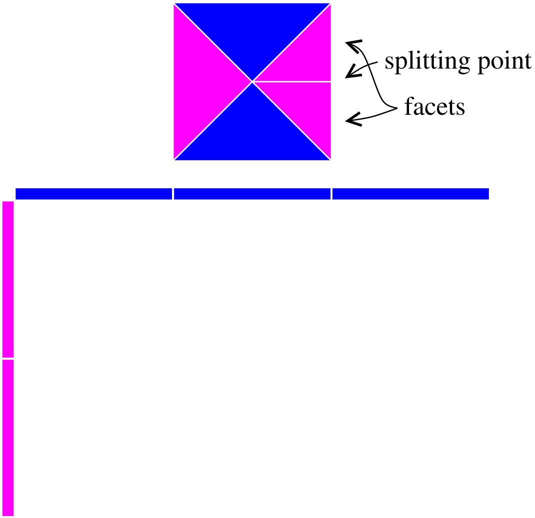
commutation right-faceted diamond ($_1\leftarrow \cdot \to_2 \subseteq \to_2 \cdot {}_1\twoheadleftarrow$)

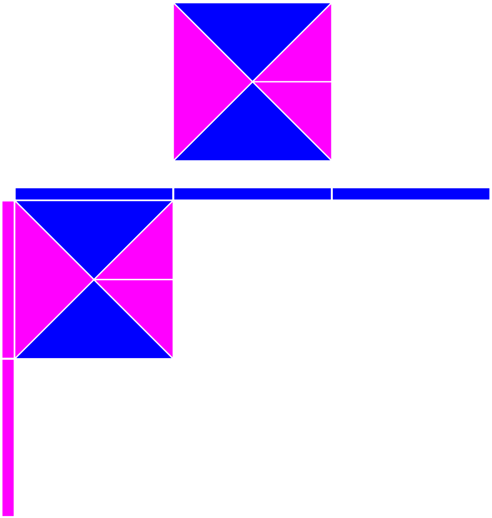
- $\lambda x.M(x) \leftarrow \lambda y.(\lambda x.M(x))\,y \to \lambda y.M(y)$
- $P\,N \leftarrow (\lambda y.P\,y)\,N \to P\,N$
- $\leftarrow \cdot \to \subseteq \to \cdot \twoheadleftarrow$

more precisely $_1\twoheadleftarrow \cdot \to_2^m \subseteq \to_2^{\leq m} \cdot {}_1\twoheadleftarrow$; valleys for critical peaks not rectangular; resolved by adjoining empty $\to_1$,$\to_2$ steps (technique 1^-)

right-faceted diamonds

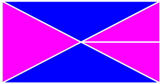


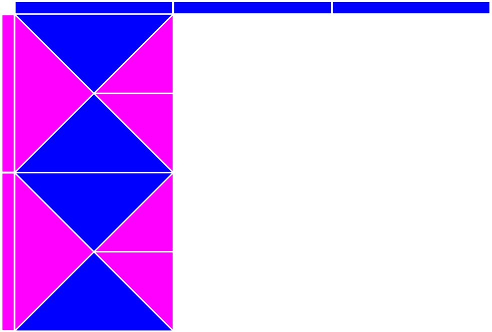
right-faceted diamonds



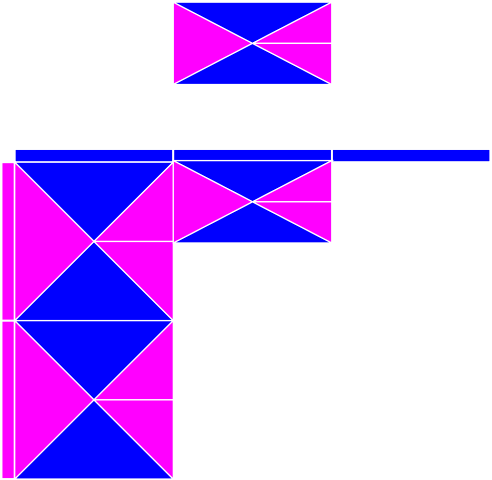
right-faceted diamonds

scale vertically to fit

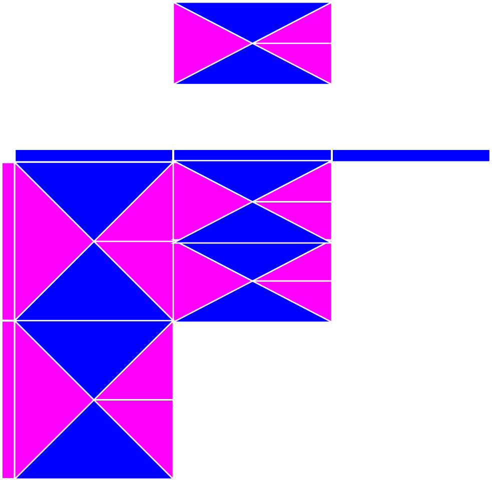




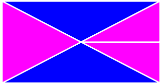
right-faceted diamonds

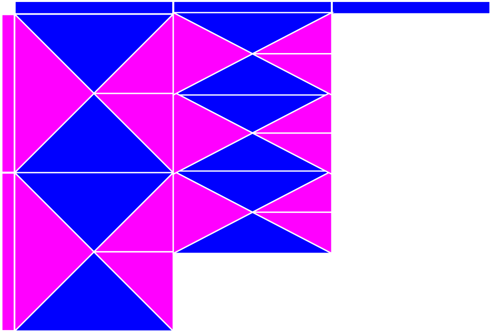


right-faceted diamonds



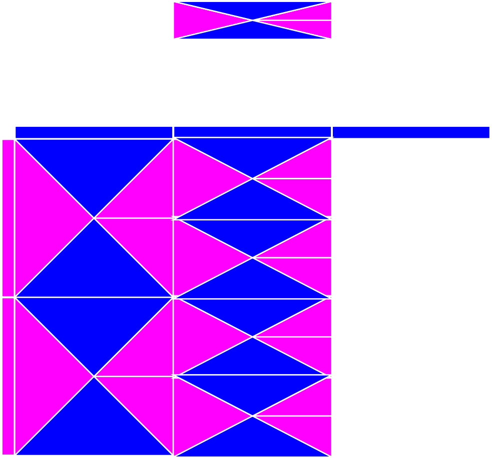
right-faceted diamonds



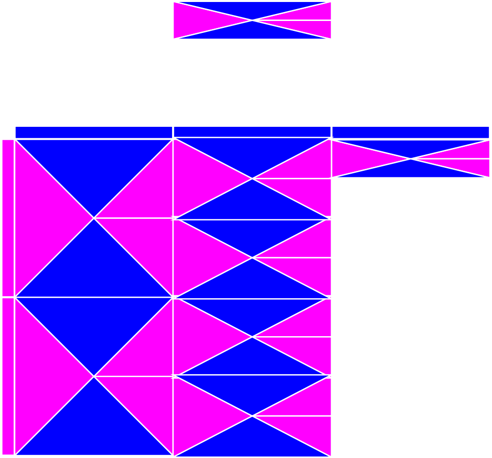


right-faceted diamonds

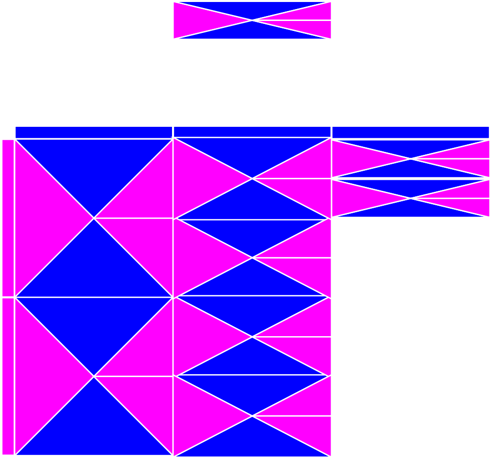
scale vertically to fit



right-faceted diamonds



right-faceted diamonds



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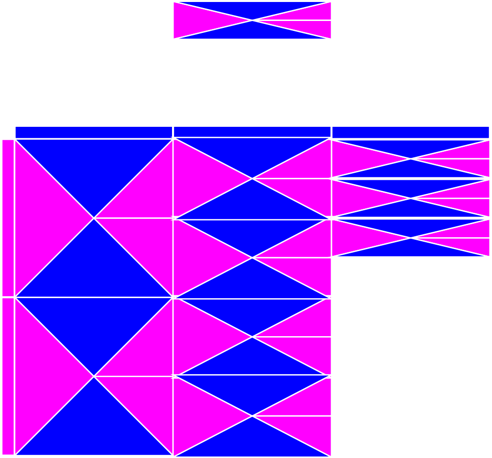
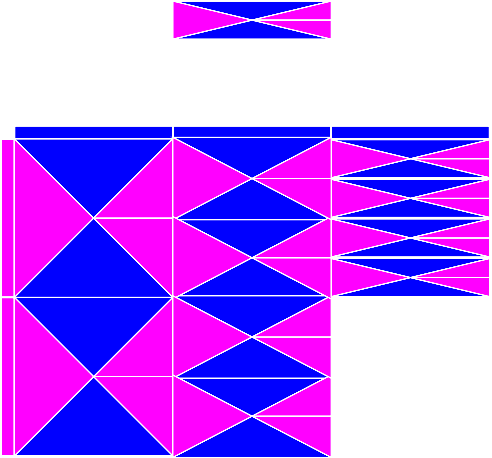


right-faceted diamonds



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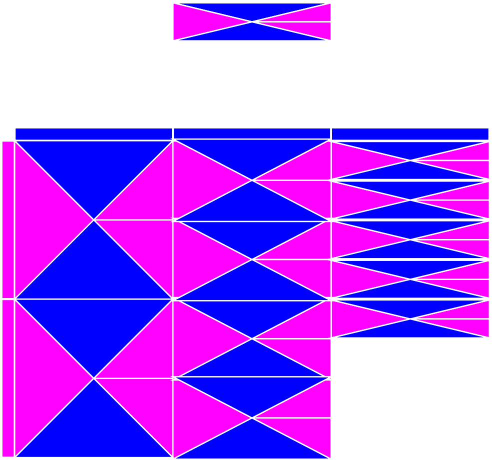

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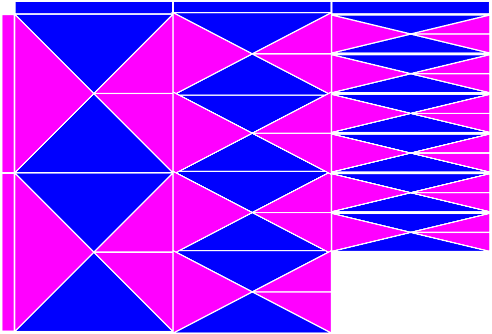

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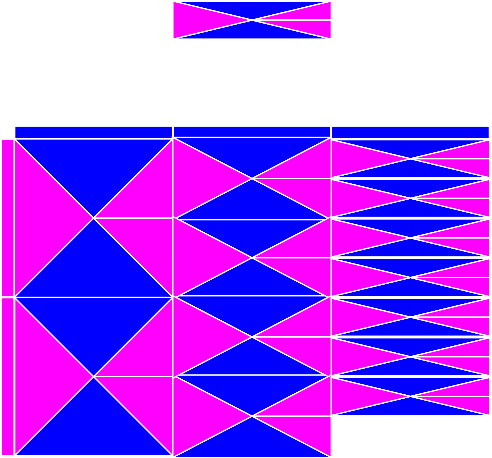

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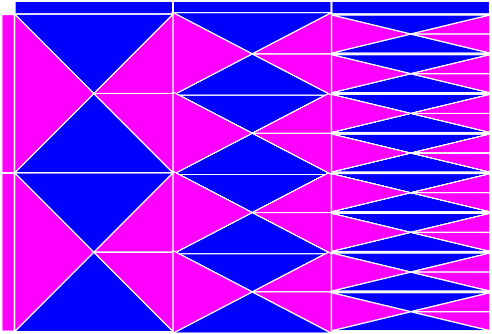
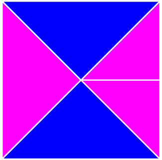
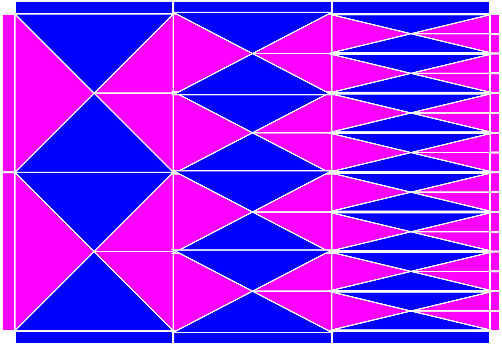


right-faceted diamonds





right-faceted diamonds

factorisation problem ($\twoheadrightarrow_1 \cdot \twoheadrightarrow_2 \subseteq \twoheadrightarrow_2 \cdot \twoheadrightarrow_1$?)

for term rewrite systems $\mathcal{T}_1$ and $\mathcal{T}_2$

- $\mathcal{T}_1 = \{P \to \lambda y.P\, y\}$ (η-expansion in λ-calculus)
- $\mathcal{T}_2 = \{(\lambda x.M(x))\, N \to M(N)\}$

2 critical peaks (between $\mathcal{T}_1^{-1}$ and $\mathcal{T}_2$); no diamonds

right-faceted diamonds

factorisation problem ($\twoheadrightarrow_1 \cdot \twoheadrightarrow_2 \subseteq \twoheadrightarrow_2 \cdot \twoheadrightarrow_1$?)

for term rewrite systems $\mathcal{T}_1$ and $\mathcal{T}_2$

- $\mathcal{T}_1 = \{P \to \lambda y.P\, y\}$
- $\mathcal{T}_2 = \{(\lambda x.M(x))\, N \to M(N)\}$

factorisation right-faceted diamond ($\to_1 \cdot \to_2 \subseteq \to_2 \cdot \twoheadrightarrow_1$)

- $\lambda x.M(x) \to \lambda y.(\lambda x.M(x))\, y \to \lambda y.M(y)$
- $P\, N \to (\lambda y.P\, y)\, N \to P\, N$
- $\to \cdot \to \subseteq \to \cdot \twoheadrightarrow$

right-faceted diamonds

factorisation problem ($\twoheadrightarrow_1 \cdot \twoheadrightarrow_2 \subseteq \twoheadrightarrow_2 \cdot \twoheadrightarrow_1$?)

for term rewrite systems $\mathcal{T}_1$ and $\mathcal{T}_2$

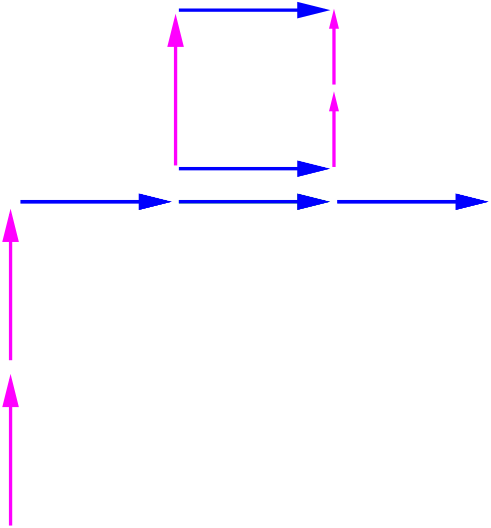
- $\mathcal{T}_1 = \{P \to \lambda y.P\,y\}$
- $\mathcal{T}_2 = \{(\lambda x.M(x))\,N \to M(N)\}$

factorisation right-faceted diamond ($\to_1 \cdot \to_2 \subseteq \to_2 \cdot \twoheadrightarrow_1$)

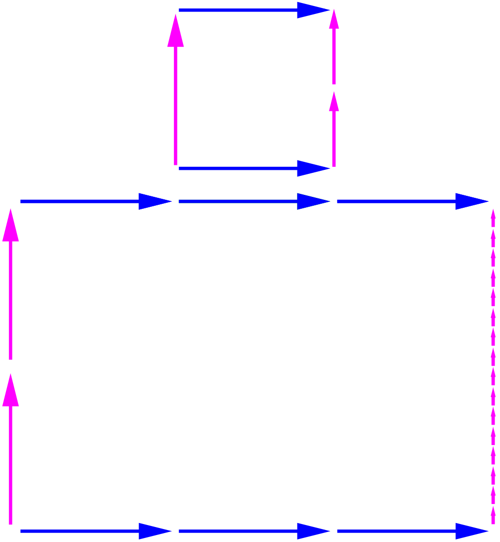
- $\lambda x.M(x) \to \lambda y.(\lambda x.M(x))\,y \to \lambda y.M(y)$
- $P\,N \to (\lambda y.P\,y)\,N \to P\,N$
- $\to \cdot \to \subseteq \to \cdot \twoheadrightarrow$

β,η^{-1}-factorisation is η,β-commutation

right-faceted diamonds



right-faceted diamonds



multi-faceted diamonds (Newman 42, de Bruijn 1978, vO 1994)

commutation problem (${}_1\twoheadleftarrow \cdot \twoheadrightarrow_2 \subseteq \twoheadrightarrow_2 \cdot {}_1\twoheadleftarrow$?)

for term rewrite systems $\mathcal{T}_1$ and $\mathcal{T}_2$

- $\mathcal{T}_1 = \{b \to a, a \to c\}$
- $\mathcal{T}_2 = \{a \to b, b \to d\}$

both right- and left-faceted diamonds

multi-faceted diamonds

commutation problem (${}_1{\twoheadleftarrow} \cdot \twoheadrightarrow_2 \subseteq \twoheadrightarrow_2 \cdot {}_1{\twoheadleftarrow}$?)

for term rewrite systems $\mathcal{T}_1$ and $\mathcal{T}_2$

- $\mathcal{T}_1 = \{b \to a, a \to c\}$
- $\mathcal{T}_2 = \{a \to b, b \to d\}$

commutation multi-faceted diamond (${}_1{\leftarrow} \cdot \to_2 \subseteq \twoheadrightarrow_2 \cdot {}_1{\twoheadleftarrow}$)

critical peaks between $\mathcal{T}_1, \mathcal{T}_2$:

- $\leftarrow \cdot \to \subseteq \leftarrow \cdot \leftarrow$(right faceted)
- $\leftarrow \cdot \to \subseteq \to \cdot \to$(left-faceted)

Counterexample $c \;{}_1{\leftarrow}\; a \;{}_1{\rightleftarrows}^2\; b \to_2 d$ to local commutation $\implies$ commutation (Kleene).

multi-faceted diamonds

commutation problem $({}_1\twoheadleftarrow \cdot \twoheadrightarrow_2 \subseteq \twoheadrightarrow_2 \cdot {}_1\twoheadleftarrow ?)$

for term rewrite systems $\mathcal{T}_1$ and $\mathcal{T}_2$

- $\mathcal{T}_1 = \{b \to a, a \to c, d \to e\}$
- $\mathcal{T}_2 = \{a \to b, b \to d, c \to e\}$

commutation multi-faceted diamond $({}_1\leftarrow \cdot \to_2 \subseteq \twoheadrightarrow_2 \cdot {}_1\twoheadleftarrow)$

critical peaks between $\mathcal{T}_1, \mathcal{T}_2$:

- $\leftarrow \cdot \to \subseteq \leftarrow \cdot \leftarrow$
- $\leftarrow \cdot \to \subseteq \to \cdot \to$

Counterexample $c\ {}_1\!\leftarrow a\ {}_1\!\rightleftarrows^2 b \to_2 d$ to local commutation $\implies$ commutation (Kleene). Adjoining $c \to_2 e\ {}_1\!\leftarrow d$ shows even if commutation holds, that need not be provable by local commutation tiling (reusing Endrullis, Grabmayer)

multi-faceted diamonds

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critical peaks between $\mathcal{T}_1, \mathcal{T}_2$:

- $\leftarrow \cdot \to \subseteq \to \cdot \leftarrow \cdot \leftarrow$ (adjoining empty $\to$-step to get rectangular tile)
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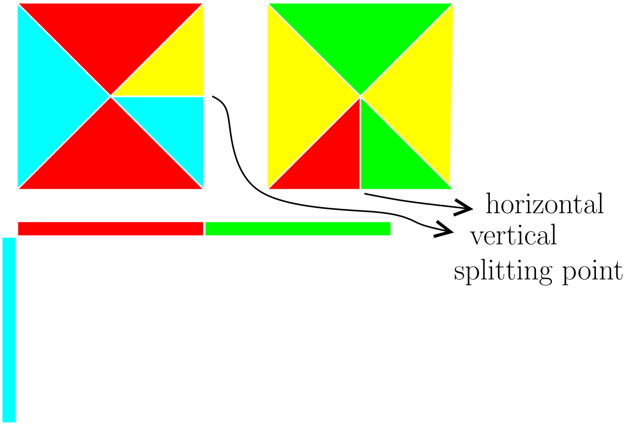
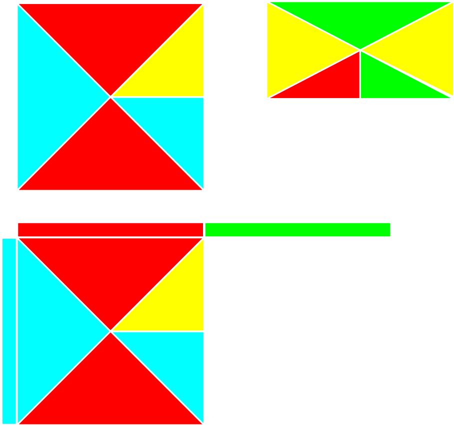
Counterexample $c \ {}_1{\leftarrow} \ a \ {}_1{\rightleftarrows}^2 \ b \to_2 d$ to local commutation $\implies$ commutation (Kleene). Adjoining $c \to_2 e \ {}_1{\leftarrow} \ d$ shows even if commutation holds, that need not be provable by local commutation tiling (reusing Endrullis, Grabmayer)

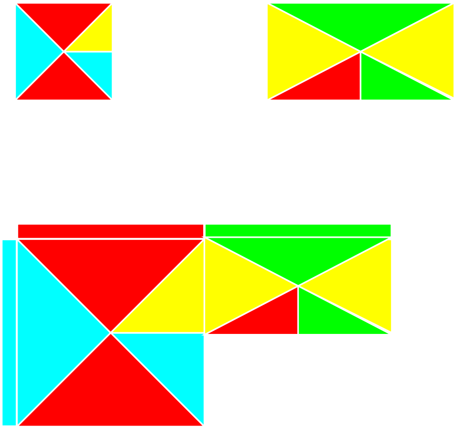
multi-faceted diamonds



multi-faceted diamonds



multi-faceted diamonds



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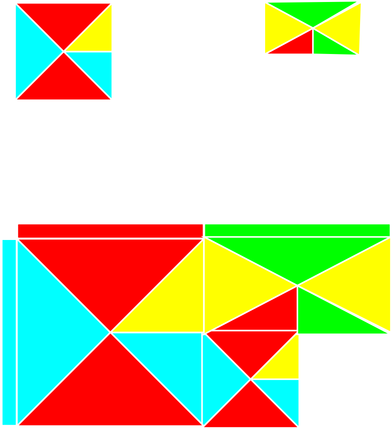

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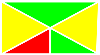

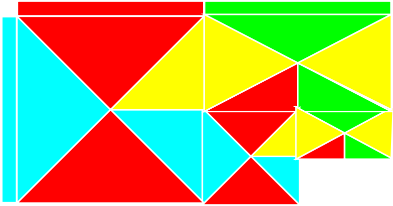

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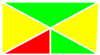
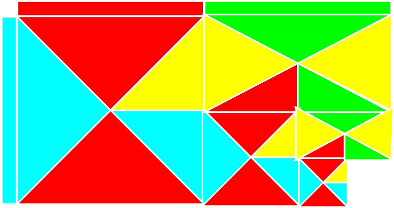
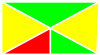
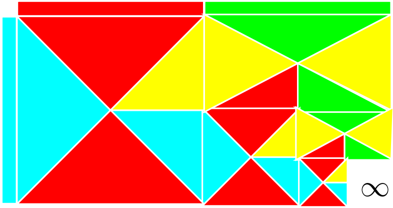




multi-faceted diamonds





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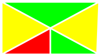

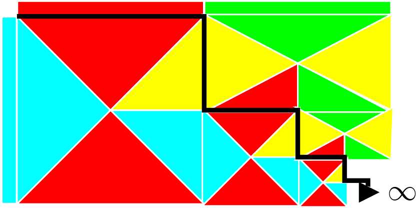

multi-faceted diamonds

splitting

- if tiling is infinite, there is an infinite reduction through infinitely many horizontal and vertical splitting points (alternatingly)

multi-faceted diamonds

splitting

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- local commutation $\Longrightarrow$ commutation, if $\rightarrow_1 \cup \rightarrow_2$ terminating (Newman 1942, Backhouse & Doornbos 1994), even if just $\rightarrow_1^+ \cdot \rightarrow_2^+$ terminating (Pous 2005)

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- extended Kleene example commuting but not terminating . . . ? Avoid splitting by adjoining certain reductions in valleys as single steps (technique 1; faceting).
- $c \leftarrow b$ (adjoined to $\mathcal{T}_1$ for $c \leftarrow \cdot \leftarrow b$)
  $a \to d$ (adjoined to $\mathcal{T}_2$ for $a \to \cdot \to d$)

multi-faceted diamonds

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- $c \leftarrow b$
  $a \to d$
- new critical peaks:
  $c \leftarrow b \to d$
  $c \leftarrow a \to d$

multi-faceted diamonds

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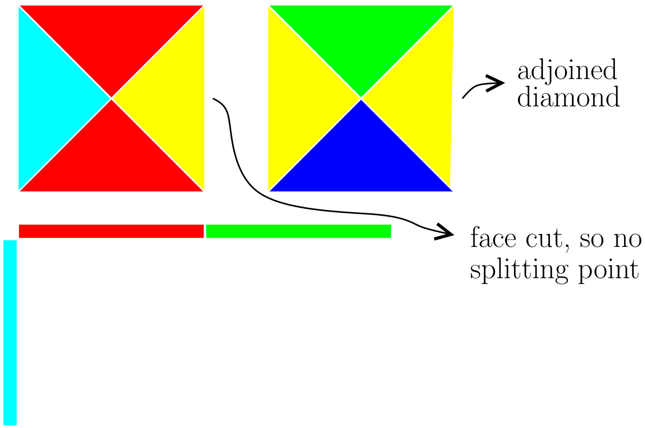
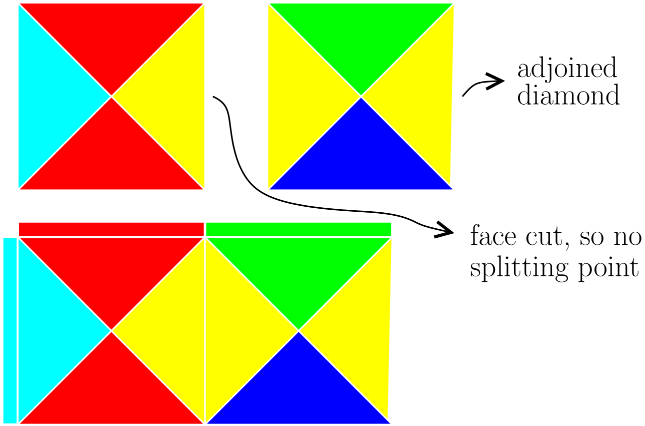
- $c \leftarrow b$
  $a \to d$
- new critical peaks:
  $c \leftarrow b \to d$ joinable by $c \to e \leftarrow d$ into diamond
  $c \leftarrow a \to d$ joinable by $c \to e \leftarrow d$ into diamond
  4 tiles in total, all (square) diamonds

multi-faceted diamonds



multi-faceted diamonds



multi-faceted diamonds

question

characterise shape of multi-faceted diamonds such that tiling always terminates?

multi-faceted diamonds

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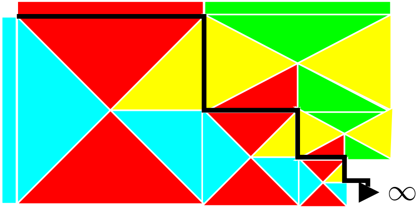
characterise shape of multi-faceted diamonds such that tiling always terminates?

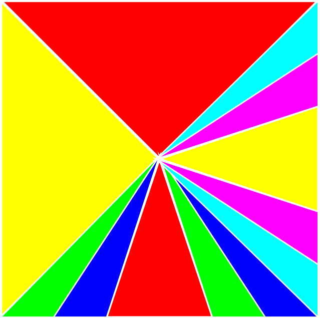


note colors alternate (between red and yellow) along infinite reduction

multi-faceted diamonds

idea

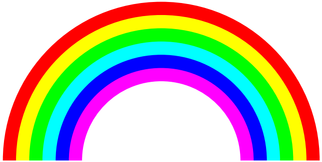
order the facets in valley below peak such that colors decrease along infinite reduction



multi-faceted diamonds

idea

order the facets in valley below peak such that colors decrease along infinite reduction

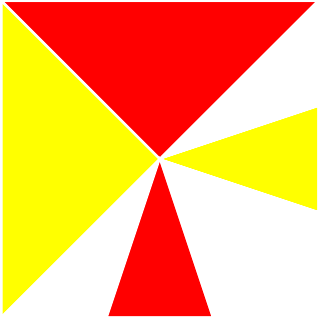


any well-founded order; here rainbow color order

multi-faceted diamonds

idea

order the facets in valley below peak such that colors decrease along infinite reduction

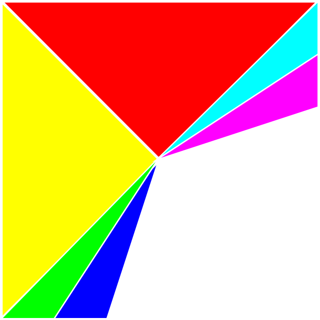


middle facet in valley same color as opposite facet in peak

multi-faceted diamonds

idea

order the facets in valley below peak such that colors decrease along infinite reduction

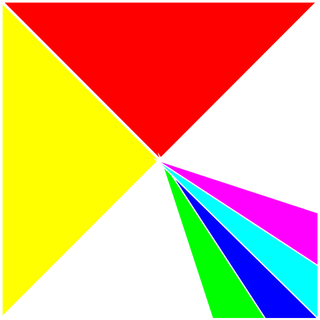


facets before middle, smaller color than adjacent facet in peak

multi-faceted diamonds

idea

order the facets in valley below peak such that colors decrease along infinite reduction

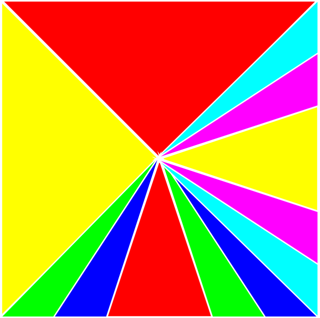


facets after middle, smaller color than either facet in peak

multi-faceted diamonds

idea

order the facets in valley below peak such that colors decrease along infinite reduction

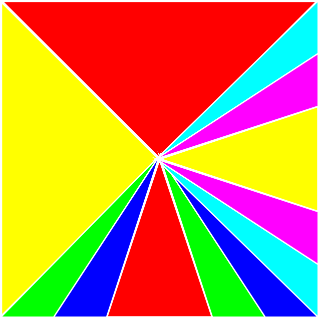


tiling peaks terminates for any set of decreasing diamonds (de Bruijn 1978)

multi-faceted diamonds

idea

order the facets in valley below peak such that colors decrease along infinite reduction



tiling peaks terminates for any set of decreasing diagrams (de Bruijn 1978)

β,η-factorisation

factorisation problem ($\twoheadrightarrow_1 \cdot \twoheadrightarrow_2 \subseteq \twoheadrightarrow_2 \cdot \twoheadrightarrow_1$?)

for term rewrite systems $\mathcal{T}_1$ and $\mathcal{T}_2$

- $\mathcal{T}_1 = \{\lambda y.P\, y \to P\}$
- $\mathcal{T}_2 = \{(\lambda x.M(x))\, N \to M(N)\}$

β,η-factorisation

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factorisation decreasing diamond?

- $(\lambda y.(\lambda x.M(x))\,y)\,N \to (\lambda x.M(x))\,N \to M(N)$ $\quad(\eta^{-1},\beta$ critical peak)
  $(\lambda y.(\lambda x.M(x))\,y)\,N \to (\lambda x.M(x))\,N \to M(N)$ $\quad$(valley of left-faceted diamond)
- $\to \cdot \to \subseteq \to \cdot \twoheadrightarrow$ $\quad$(non-critical peaks; right faceted diamonds)

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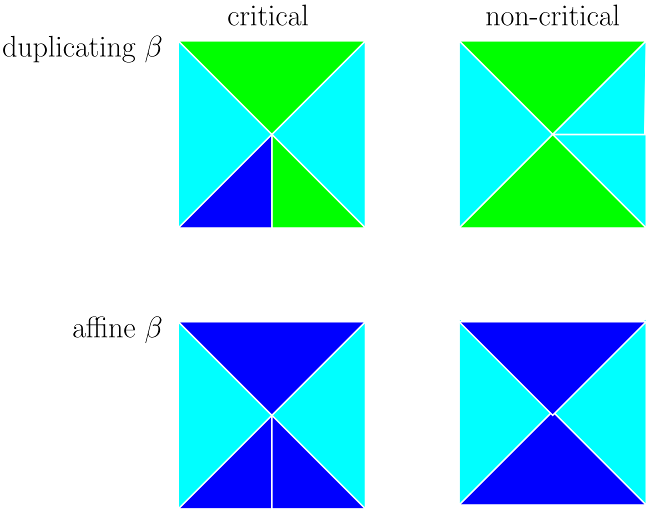
  $(\lambda y.(\lambda x.M(x))\,y)\,N \to (\lambda x.M(x))\,N \to M(N)$
- $\to \cdot \to \subseteq \to \cdot \twoheadrightarrow$

first β in critical valley is specialisation of β (technique 2; Hirokawa et al. 2019)

- $(\lambda x.M(x))\,N \to M(N)$ if x occurs ≤ 1 times in M
- $(\lambda x.M(x))\,N \to M(N)$ if x occurs > 1 times in M

renders al diamonds decreasing

β,η-factorisation



spine,vertebrae-factorisation

factorisation problem ($\twoheadrightarrow_1 \cdot \twoheadrightarrow_2 \subseteq \twoheadrightarrow_2 \cdot \twoheadrightarrow_1$?)

for rewrite systems $\mathcal{T}_1$ and $\mathcal{T}_2$ on the set of λ-terms

- $\mathcal{T}_1 = \rightarrow$ may contract any β-redex at vertebrae position $(\notin 1^*)$
- $\mathcal{T}_2 = \rightarrow$ may contract any β-redex at spine position $(\in 1^*)$

note $\rightarrow_\beta = \rightarrow \cup \rightarrow$

spine,vertebrae-factorisation

factorisation problem ($\twoheadrightarrow_1 \cdot \twoheadrightarrow_2 \subseteq \twoheadrightarrow_2 \cdot \twoheadrightarrow_1$?)

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- $\mathcal{T}_1 = \rightarrow$ may contract any β-redex at vertebrae position
- $\mathcal{T}_2 = \rightarrow$ may contract any β-redex at spine position

factorisation decreasing diamond for $\rightarrow, \rightarrow$?

- no critical peaks (→ cannot create →; spine closed under prefix)
- $\rightarrow \cdot \rightarrow \subseteq \rightarrow \cdot \twoheadrightarrow_\beta$ (non-critical peak; → cannot replicate →)

note $\twoheadrightarrow_\beta$ here is development of residuals of → after → (both from source)

spine,vertebrae-factorisation

factorisation problem ($\twoheadrightarrow_1 \cdot \twoheadrightarrow_2 \subseteq \twoheadrightarrow_2 \cdot \twoheadrightarrow_1$?)

for rewrite systems $\mathcal{T}_1$ and $\mathcal{T}_2$ on the set of λ-terms

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factorisation decreasing diamond for $\rightarrow, \rightarrow$?

- no critical peaks
- $\rightarrow \cdot \rightarrow \subseteq \rightarrow \cdot \twoheadrightarrow_\beta$ (non-critical peak; $\rightarrow$ cannot replicate $\rightarrow$)

example

$(\lambda x.x\,x)\,((\lambda y.y)\,z) \rightarrow (\lambda x.x\,x)\,z \rightarrow z\,z$ factorises to
$(\lambda x.x\,x)\,((\lambda y.y)\,z) \rightarrow (\lambda y.y)\,z\,((\lambda y.y)\,z) \rightarrow z\,((\lambda y.y)\,z) \rightarrow z\,z$
may yield multiple $\rightarrow, \rightarrow$-steps $\Longrightarrow$ choose to facet $\rightarrow$-developments as $\rightarrow$

spine,vertebrae-factorisation

factorisation problem ($\twoheadrightarrow_1 \cdot \twoheadrightarrow_2 \subseteq \twoheadrightarrow_2 \cdot \twoheadrightarrow_1$?)

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factorisation decreasing diamond for $\rightarrow, \rightarrow$?

- still no critical peaks
- $\rightarrow \cdot \rightarrow \subseteq \rightarrow \cdot \twoheadrightarrow_\beta \subseteq \rightarrow \cdot \twoheadrightarrow \cdot \rightarrow$ (non-critical peak; is decreasing diamond)

development of $\rightarrow$-step is $\rightarrow$-reduction (cf. Melliès' segmentation property)

spine,vertebrae-factorisation

factorisation problem ($\twoheadrightarrow_1 \cdot \twoheadrightarrow_2 \subseteq \twoheadrightarrow_2 \cdot \twoheadrightarrow_1$?)

for rewrite systems $\mathcal{T}_1$ and $\mathcal{T}_2$ on the set of λ-terms

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factorisation decreasing diamond for $\rightarrow, \rightarrow$?

- still no critical peaks
- $\rightarrow \cdot \rightarrow \subseteq \rightarrow \cdot \twoheadrightarrow_\beta \subseteq \rightarrow \cdot \twoheadrightarrow \cdot \rightarrow$ (non-critical peak; is decreasing diamond)

adaptations

same critical peak analysis works for head,internal-factorisation for β-reduction:

- head-steps have unique origin along internal steps (head-positions closed under prefix; if rhs of step overlaps/is above head-redex then step is itself head)
- developing a set of internal redexes yields internal reduction

self-commutation of some term rewrite system

some term rewrite system

- three rules of which the 1st is (self-)replicating, the other two $\rightarrow$, $\rightarrow$ linear

self-commutation of some term rewrite system

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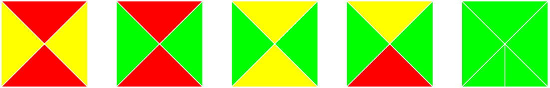
- for non-critical peaks facet developments of 1st as $\rightarrow$, ordered above $\rightarrow$,$\rightarrow$-steps

self-commutation of some term rewrite system

some term rewrite system

- three rules of which the 1st is (self-)replicating, the other two $\to$, $\to$ linear
- for non-critical peaks facet developments of 1st as $\to$, ordered above $\to$,$\to$-steps
- for critical peaks:

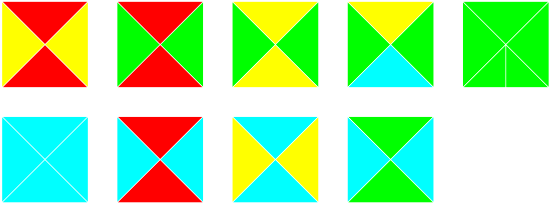


fourth diagram then not decreasing, but only linear specialisation $\to$ of $\to$ needed

self-commutation of some term rewrite system

some term rewrite system

- three rules of which the 1st is (self-)replicating, the other two $\to$, $\to$ linear
- for non-critical peaks facet developments of 1st as $\to$, ordered above $\to$,$\to$-steps
- critical peaks after adjoining linear specialisation $\to$:



fifth diagram not decreasing, but $\to \cup \to \cup \to$ terminating (SOL, Hamana 2020)

self-commutation of some term rewrite system

some term rewrite system

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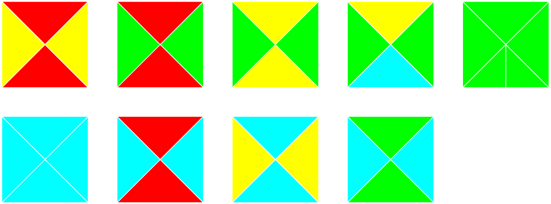

  fifth diagram not decreasing, but $\to \cup \to \cup \to$ terminating (SOL, Hamana 2020)
- source labelling these (all still ordered below $\to$), all decreasing $\implies$ confluence

take-aways

- commutation = factorisation, up to symmetry

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- for structured (string, term, . . .) rewrite systems, analysed via critical peaks, i.e. overlaps between left- respectively right-hand sides of 1st, left-hand sides of 2nd

take-aways

- commutation $=$ factorisation, up to symmetry
- for structured (string, term, ...) rewrite systems, analysed via critical peaks, i.e. overlaps between left- respectively right-hand sides of 1st, left-hand sides of 2nd
- two techniques for making diagrams decreasing
  1. faceting: adjoining certain reductions in valleys as rules (parallel steps, developments for term rewriting, left-divisors of Garside-element for braids, empty reductions)
  2. specialisation: adjoining rules in context,substitution as rules

take-aways

- commutation = factorisation, up to symmetry
- for structured (string, term, . . .) rewrite systems, analysed via critical peaks, i.e. overlaps between left- respectively right-hand sides of 1st, left-hand sides of 2nd
- two techniques for making diagrams decreasing
  1. faceting: adjoining certain reductions in valleys as rules
  2. specialisation: adjoining rules in context,substitution as rules
- diagrammatic: every peak filled by local commutation diagrams if decreasing

take-aways from Newman 1942

- that rewriting is not about relations, but steps
- his lemma and its homotopic strengthening: for terminating and locally confluent rewrite system all diagrams (cycles) deformable into the empty diagram (cf. Squier 1987, Kraus & von Raumer 2020)
- diamond property and random descent (Toyama 1992, vO 2007, T & vO 2016)
- axiomatic residuals (Hindley, Glauert & Khasidashvili, Melliès, Terese) (α-equivalence error in application to λ-calculus; but expect it applies to TRSs)
- interest in least upperbounds (left to future work; cf. orthogonality in term rewriting or braids; faceting by least way to extend co-initial steps)