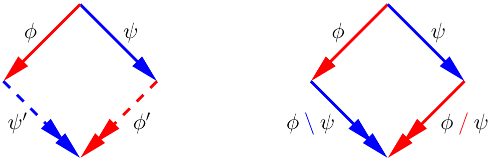
Residuation = Skolemised Confluence

Vincent van Oostrom

Skolemising local confluence

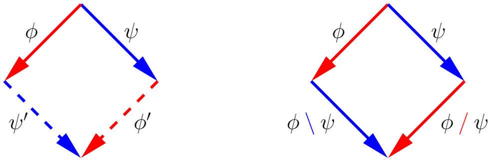


Definition (local confluence)

- $\forall$ peak $b \leftarrow a \rightarrow c$ of steps $\exists$ valley $b \twoheadrightarrow d \twoheadleftarrow c$ of reductions, for some d
- $\forall$ peak of steps ϕ, ψ, have that $\phi \setminus \psi, \phi \,/\, \psi$ is composable valley of reductions

Skolem functions $\setminus, /$ from (co-initial) steps to (co-final) reductions: residuations
write $C(\phi, \psi, \phi \setminus \psi, \phi \,/\, \psi)$ to denote reductions constitute confluence diagram

Skolemising the diamond property



Definition (local confluence)

- $\forall$ peak $b \leftarrow a \rightarrow c$ of steps $\exists$ valley $b \rightarrow d \leftarrow c$ of steps, for some d
- $\forall$ peak of steps ϕ, ψ, have that $\phi \setminus \psi, \phi / \psi$ is composable valley of steps

Skolem functions $\setminus$, $/$ from (co-initial) steps to (co-final) steps: residuations
write $C(\phi, \psi, \phi \setminus \psi, \phi / \psi)$ to denote reductions constitute confluence diagram

From two residuations $\backslash, /$ to a single symmetric one $|$

Definition

residuations $\backslash, /$ are symmetric if $\phi \backslash \psi = \psi / \phi$

Remark

a priori diamond property / (local) confluence need not entail symmetry

- for $\phi, \psi : a \to a$, may set $\phi \backslash \psi := \psi \backslash \phi := \phi$ and $\phi / \psi := \psi / \phi := \psi$
- if e.g. $\leftarrow \cdot \to \subseteq \twoheadrightarrow \cdot \stackrel{=}{\leftarrow}$ then confluence typically not symmetric (strong confluence; Huet)
- orthogonal term rewrite system, diamond of multisteps $\multimap$ is symmetric (Rosen, Tait–Martin-Löf, Klop proof)

From two residuations $\backslash, /$ to a single symmetric one $|$

Definition

residuations $\backslash, /$ are symmetric if $\phi \backslash \psi = \psi / \phi$

Question

does a symmetric residuation always exist?

From two residuations $\backslash, /$ to a single symmetric one $|$

Definition

residuations $\backslash, /$ are symmetric if $\phi \backslash \psi = \psi / \phi$

Lemma

residuations may be assumed to be symmetric

Proof.

by symmetrisation given $\backslash, /$, assuming a total order $\leq$ on (co-initial) steps define $\phi \mid \psi := \phi / \psi$ if $\phi \leq \psi$ and $\psi \backslash \phi$ otherwise. $\square$

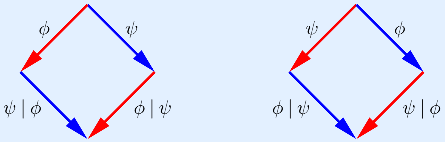
From two residuations $\backslash$, $/$ to a single symmetric one $|$

Lemma

residuations may be assumed to be symmetric

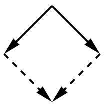
Proof.

by symmetrisation given $\backslash$, $/$, assuming a total order $\leq$ on (co-initial) steps define $\phi \mid \psi := \phi \,/\, \psi$ if $\phi \leq \psi$ and $\psi \backslash \phi$ otherwise. then if $\phi \leq \psi$:



□

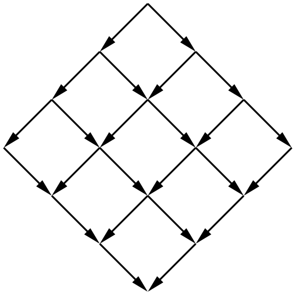
From diamond to confluence (diamond of $\twoheadrightarrow$) by tiling



Theorem (Newman)

if $\leftarrow \cdot \rightarrow \subseteq \rightarrow \cdot \leftarrow$ *then* $\twoheadleftarrow \cdot \twoheadrightarrow \subseteq \twoheadrightarrow \cdot \twoheadleftarrow$

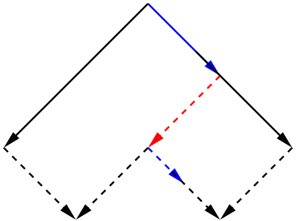
From diamond to confluence by tiling



Theorem (Newman)

if $\to$ has diamond property then so does $\twoheadrightarrow$ (confluence); results unique

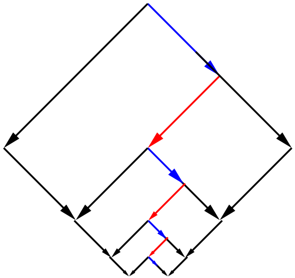
From local confluence to confluence by tiling



Theorem (Newman)

if $\leftarrow \cdot \rightarrow \subseteq \twoheadrightarrow \cdot \twoheadleftarrow$ *then* $\twoheadleftarrow \cdot \twoheadrightarrow \subseteq \twoheadrightarrow \cdot \twoheadleftarrow$, *if* $\rightarrow$ *terminating*

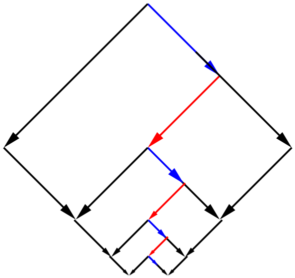
From local confluence to confluence by tiling



Theorem (Newman)

if $\to$ is locally confluent then confluent, if no infinite blue-red paths

From local confluence to confluence by tiling



Remark (on partiality)

residuation on steps, in general extends to partial residuation on reductions

Extending residuation from steps to reductions by tiling

Idea (Newman, Klop, . . .)

try to stepwise transform a peak of reductions into a valley of reductions by replacing local peaks of local confluence / diamond by valley

Extending residuation from steps to reductions by tiling

Idea (Newman, Klop, . . .)

try to stepwise transform a peak of reductions into a valley of reductions by replacing local peaks of local confluence / diamond by valley

Questions

- what are intermediate stages?

Extending residuation from steps to reductions by tiling

Idea (Newman, Klop, . . .)

try to stepwise transform a peak of reductions into a valley of reductions by replacing local peaks of local confluence / diamond by valley

Questions

- what are intermediate stages?
  conversions, compositions $(\cdot)$ of forward and backward $({}^{-1})$ steps

Extending residuation from steps to reductions by tiling

Idea (Newman, Klop, . . .)

try to stepwise transform a peak of reductions into a valley of reductions by replacing local peaks of local confluence / diamond by valley

Questions

- what are intermediate stages?
  conversions, compositions ($\cdot$) of forward and backward ($^{-1}$) steps
- what are transformation rules?

Extending residuation from steps to reductions by tiling

Idea (Newman, Klop, . . .)

try to stepwise transform a peak of reductions into a valley of reductions by replacing local peaks of local confluence / diamond by valley

Questions

- what are intermediate stages?
  conversions, compositions $(\cdot)$ of forward and backward $(\,^{-1})$ steps
- what are transformation rules?
  $\phi^{-1} \cdot \psi \Rightarrow (\phi \setminus \psi) \cdot (\phi \,/\, \psi)^{-1}$ for (co-inital) steps ϕ, ψ

Extending residuation from steps to reductions by tiling

Idea (Newman, Klop, . . .)

try to stepwise transform a peak of reductions into a valley of reductions by replacing local peaks of local confluence / diamond by valley

Questions

- what are intermediate stages?
  conversions, compositions $(\cdot)$ of forward and backward $({}^{-1})$ steps
- what are transformation rules?
  $\phi^{-1} \cdot \psi \Rightarrow (\phi \setminus \psi) \cdot (\phi / \psi)^{-1}$ for (co-inital) steps ϕ, ψ
- what are $\phi \setminus \psi, \phi / \psi$ for (co-initial) reductions ϕ, ψ?

Extending residuation from steps to reductions by tiling

Idea (Newman, Klop, . . .)

try to stepwise transform a peak of reductions into a valley of reductions by replacing local peaks of local confluence / diamond by valley

Questions

- what are intermediate stages?
  conversions, compositions $(\cdot)$ of forward and backward $({}^{-1})$ steps
- what are transformation rules?
  $\phi^{-1} \cdot \psi \Rightarrow (\phi \setminus \psi) \cdot (\phi / \psi)^{-1}$ for (co-inital) steps ϕ, ψ
- what are $\phi \setminus \psi, \phi / \psi$ for (co-initial) reductions ϕ, ψ?
  left, right reduction of $\Rightarrow$-normal form (valley) of $\phi^{-1} \cdot \psi$

Well-behavedness of extending residuation by tiling

Lemma

- *extensions of $\backslash$, $/$ to reductions are functions (partial if $\Rightarrow$ non-terminating) by $\Rightarrow$-reductions to normal form having same length (random descent* ⩔*)*

Well-behavedness of extending residuation by tiling

Lemma

- *extensions of $\backslash, /$ to reductions are functions (partial if $\Rightarrow$ non-terminating) by $\Rightarrow$-reductions to normal form having same length (random descent)*
- *extension preserves being symmetric by $^{-1}$ being an involution on conversions and $\Rightarrow$-steps*

Well-behavedness of extending residuation by tiling

Lemma

- *extensions of* $\backslash, /$ *to reductions are functions (partial if* $\Rightarrow$ *non-terminating)* *by* $\Rightarrow$*-reductions to normal form having same length (random descent)*
- *extension preserves being symmetric* *by* $^{-1}$ *being an involution on conversions and* $\Rightarrow$*-steps*
- *for any (countable) confluent rewrite system there exist residuations on peaks of steps that extend by tiling to residuations on peaks of reductions* *by completeness of decreasing diagrams (Ken Mano,)*

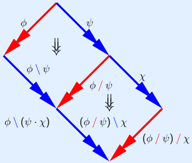
Laws for residuation of reductions by tiling

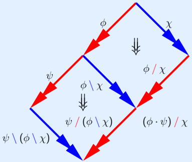
Lemma

$$\begin{array}{rcl} \phi / \varepsilon & = & \phi \\ \phi \backslash \varepsilon & = & \varepsilon \\ \phi / (\psi \cdot \chi) & \simeq & (\phi / \psi) / \chi \\ \phi \backslash (\psi \cdot \chi) & \simeq & (\phi \backslash \psi) \cdot ((\phi / \psi) \backslash \chi) \end{array}$$

$$\begin{array}{rcl} \varepsilon \backslash \phi & = & \phi \\ \varepsilon / \phi & = & \varepsilon \\ (\phi \cdot \psi) \backslash \chi & \simeq & \psi \backslash (\phi \backslash \chi) \\ (\phi \cdot \psi) / \chi & \simeq & (\phi / \chi) \cdot (\psi / (\phi \backslash \chi)) \end{array}$$

Proof by appropriate tiling.





□

Implementing residuation for reductions by recursion

Recursive implementation

$$(\phi \cdot \psi) \,/\, (\chi \cdot \omega) \;:=\; ((\phi \,/\, \chi) \cdot (\psi \,/\, (\phi \setminus \chi))) \,/\, \omega$$
$$(\phi \cdot \psi) \setminus (\chi \cdot \omega) \;:=\; \psi \setminus ((\phi \setminus \chi) \cdot ((\phi \,/\, \chi) \setminus \omega))$$

for steps ϕ, χ and (non-empty) reductions ψ, ω

Implementing residuation for reductions by recursion

Recursive implementation

$$
\begin{aligned}
(\phi \cdot \psi) \,/\, (\chi \cdot \omega) &:= ((\phi \,/\, \chi) \cdot (\psi \,/\, (\phi \setminus \chi))) \,/\, \omega \\
(\phi \cdot \psi) \setminus (\chi \cdot \omega) &:= \psi \setminus ((\phi \setminus \chi) \cdot ((\phi \,/\, \chi) \setminus \omega))
\end{aligned}
$$

for steps ϕ, χ and (non-empty) reductions ψ, ω

Remark

for symmetric residuation $|$ single clause suffices:

$$
(\phi \cdot \psi) \,|\, (\chi \cdot \omega) := ((\phi \,|\, \chi) \cdot (\psi \,|\, (\chi \,|\, \phi))) \,|\, \omega
$$

Intermediate conclusion

1. residuation = function obtained by skolemising confluence
   allows algebraic reasoning, just like having composition (categories) does

Intermediate conclusion

1. residuation = function obtained by skolemising confluence
2. residuation : composition (steps) = monus : addition (natural numbers)

Intermediate conclusion

1. residuation = function obtained by skolemising confluence
2. residuation : composition = monus : addition
3. residuation may be total where composition partial
   e.g. on (multi-/parallel)steps; allows equational reasoning

Intermediate conclusion

1. residuation = function obtained by skolemising confluence
2. residuation : composition = monus : addition
3. residuation may be total where composition partial
4. “Tait–Martin-Löf proof of confluence for $\lambda\beta$ is easier than with residuals” nonsense; can’t have one without the other; also concretely false for $\lambda\beta$:

ζ	ξ	$\zeta \mid \xi$
$\mathsf{app}(\mathsf{abs}(\lambda x.\phi(x)), \psi)$	$\beta(\lambda x.\chi(x), \omega)$	$(\lambda x.\phi(x) \mid \chi(x))\,(\psi \mid \omega)$
$\beta(\lambda x.\phi(x), \psi)$	$\beta(\lambda x.\chi(x), \omega)$	$(\lambda x.\phi(x) \mid \chi(x))\,(\psi \mid \omega)$
$\beta(\lambda x.\phi(x), \psi)$	$\mathsf{app}(\mathsf{abs}(\lambda x.\chi(x)), \omega)$	$\beta(\lambda x.(\phi(x) \mid \chi(x)), \psi \mid \omega)$

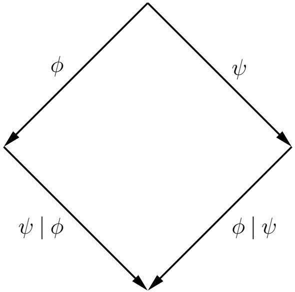
Intermediate conclusion

1. residuation = function obtained by skolemising confluence
2. residuation : composition = monus : addition
3. residuation may be total where composition partial
4. “TML is more easily expressed and shown using residuation on multisteps”
5. links to many areas: residuated lattices, pushouts and fractions in categories, linguistics (Lambek’s $\backslash$ and $/$), groupoid theory (Dehornoy et al.), probability theory (conditional probability $|$ is a residuation), ...

Intermediate conclusion

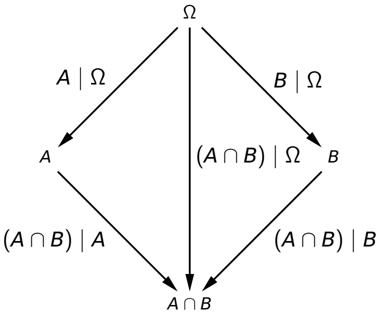
1. residuation = function obtained by skolemising confluence
2. residuation : composition = monus : addition
3. residuation may be total where composition partial
4. “TML is more easily expressed and shown using residuation on multisteps”
5. links to many areas

Bayes' Theorem via residuation of events



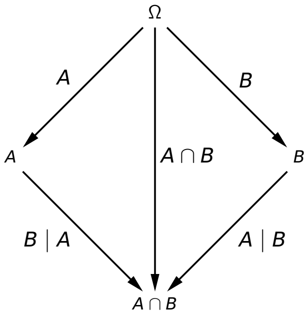
diamond property for symmetric residuation $\mid$

Bayes' Theorem via residuation of events



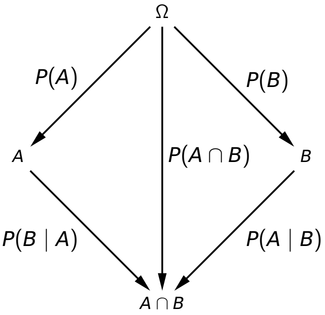
rewrite system: objects event spaces, steps $A \mid B$ (A given B; restricting B to A)

Bayes' Theorem via residuation of events



restriction is a residuation

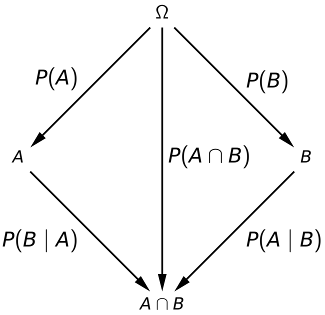
Bayes' Theorem via residuation of events



Theorem (Bayes)

$$P(A) \cdot P(B \mid A) = P(A \cap B) = P(B \cap A) = P(B) \cdot P(A \mid B)$$

Bayes' Theorem via residuation of events



$P((A \mid B) \mid (C \mid B)) = P((A \mid C) \mid (B \mid C))$ makes sense (and is true)

From residuation to residual system

Idea

- confluence = existence of upper bounds = having a residuation (viewing $\twoheadrightarrow$ as (quasi-)order; already by Newman)

From residuation to residual system

Idea

- confluence = existence of upper bounds = having a residuation
- orthogonality = existence of lubs = residual system (residuation + laws) (Newman, Plotkin, Stark, Melliès, and de Vrijer,. . .)

From residuation to residual system

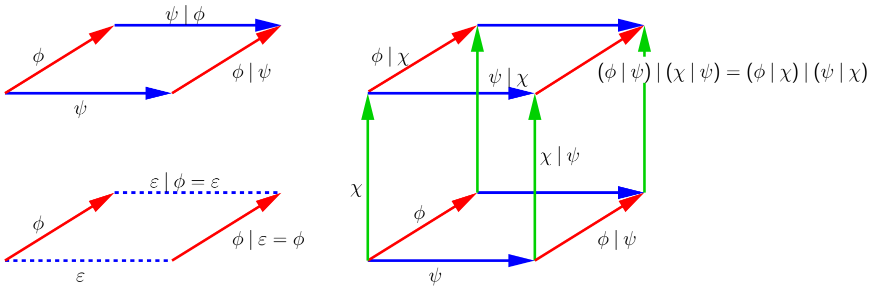
Idea

- confluence = existence of upper bounds = having a residuation
- orthogonality = existence of lubs = residual system (residuation + laws)

Definition (residual system (typed) laws for (symmetric) residuation $|$)

$$
\begin{aligned}
\phi \,|\, \varepsilon &= \phi \\
\varepsilon \,|\, \phi &= \varepsilon \\
\phi \,|\, \phi &= \varepsilon \\
(\phi \,|\, \psi) \,|\, (\chi \,|\, \psi) &= (\phi \,|\, \chi) \,|\, (\psi \,|\, \chi)
\end{aligned}
$$

Visualising the residual laws

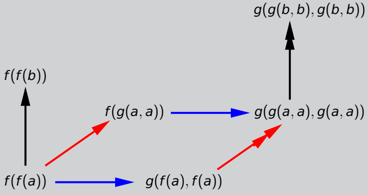


$(\phi \mid \psi) \mid (\chi \mid \psi) = (\phi \mid \chi) \mid (\psi \mid \chi)$ is cube law (Lévy); involves 3-peak ϕ, ψ, χ

Failure of cube property by tiling with local confluences

Example (for an orthogonal (duplicating) TRS)

3-peak from $f(f(a))$ in OTRS with rules $f(x) \to g(x,x)$ and $a \to b$



- $g(g(a,a),g(a,a)) \twoheadrightarrow g(g(b,b),g(a,a)) \twoheadrightarrow g(g(b,b),g(b,b))$, tiling front–right
- $g(g(a,a),g(a,a)) \twoheadrightarrow g(g(b,a),g(b,a)) \twoheadrightarrow g(g(b,b),g(b,b))$, tiling left–back

From local cubes to bricks by bricklaying

Question

if have cubes for 3-peaks of steps, can any 3-peak of reductions be completed into cube of reductions (3-confluence)
do residual systems extend from steps to reductions?

From local cubes to bricks by bricklaying

Question

if have cubes for 3-peaks of steps, can any 3-peak of reductions be completed into cube of reductions (3-confluence)
do residual systems extend from steps to reductions?

Answer

- if all faces of cubes are diamonds, then yes by stacking cubes
  the 3D version of confluence by tiling
- if faces can be (local) confluences, then bricklaying may not terminate . . .

From local cubes to bricks by bricklaying

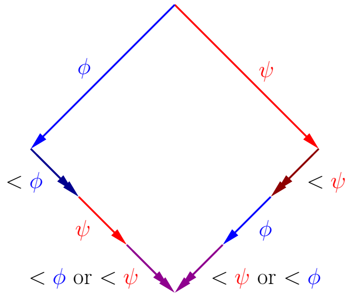
Question

if have cubes for 3-peaks of steps, can any 3-peak of reductions be completed into cube of reductions (3-confluence)
do residual systems extend from steps to reductions?

Answer

- if all faces of cubes are diamonds, then yes by stacking cubes
  the 3D version of confluence by tiling
- if faces can be (local) confluences, then bricklaying may not terminate
  decreasing diagrams to the rescue

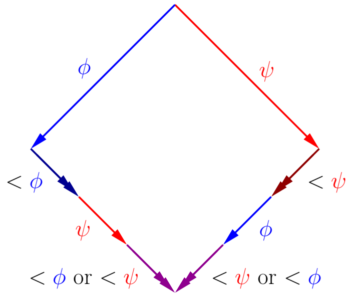
Confluence by decreasing diagrams



Definition (family $(\to_i)_{i \in I}$ of rewrite systems, $<$ well-founded order on I)

$\forall$ co-initial steps ϕ, ψ, $\exists$ co-final reductions, giving decreasing diagram

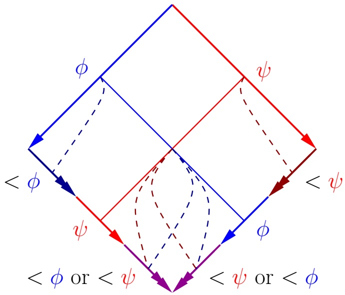
Confluence by decreasing diagrams



Definition (family $(\to_i)_{i \in I}$ of rewrite systems, $<$ well-founded order on I)

$\forall$ co-initial steps ϕ, ψ, $\exists$ co-final reductions, giving decreasing diagram

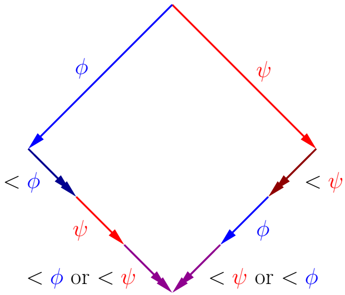
Confluence by decreasing diagrams



Definition ($\rightarrow := \bigcup_{i \in I} \rightarrow_i$ for decreasing family)

idea: steps (strictly) decrease along (dashed) traces

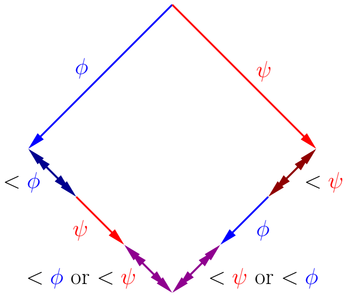
Confluence by decreasing diagrams



Theorem (based on Newman, Huet, Hindley, De Bruijn)

if $\rightarrow$ has decreasing family then confluent

Confluence by decreasing diagrams



Theorem (based on Newman, Winkler & Buchberger, Pous)

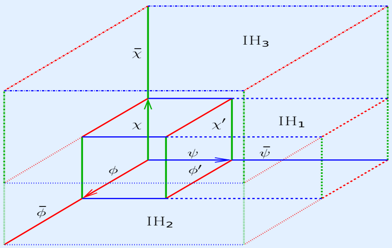
if $\rightarrow$ has decreasing family then confluent

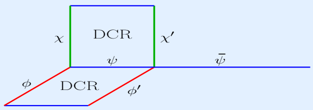
Cube property by 3-decreasing cubes

Lemma

local 3-decreasingness entails 3-decreasingness

Proof.





□

Mediate conclusion

- skolemisation of confluence into residuation makes it algebraic
- diamond : tiling = cube : bricklaying (in paper)
  tiling $\Rightarrow$ transforms conversions; bricklaying $\Rrightarrow$ transforms bed-graphs
  planar graph rewriting (non-trivial; cf. 4-colour theorem by Gonthier)
- both confluence / upper bounds / residuation and orthogonality / least upper bounds / residual systems of interest